Lab 15
Taylor Polynomials

Objectives
1. To develop an understanding for error bound, error term, and interval of convergence.
2. To visualize the convergence of the Taylor polynomials $P_n(x)$ to $f(x)$ as $n$ increases.

Taylor's Formula
Let $f$ be a function such that $f$ and its first $n$ derivatives are continuous on the closed interval $I$ and let $f^{(n+1)}(x)$ exist on the interior of $I$. Then for $c$ and $x$ in $I$, there is a number $\xi$ strictly between $c$ and $x$ such that

$$f(x) = P_n(x) + R_n(x)$$

where $P_n(x)$ is the $n$th degree Taylor Polynomial given by

$$P_n(x) = f(c) + \frac{f'(c)}{1!}(x-c) + \frac{f''(c)}{2!}(x-c)^2 + \cdots + \frac{f^{(n)}(c)}{n!}(x-c)^n$$

and $R_n(x)$ is Lagrange's form of the remainder term

$$R_n(x) = \frac{f^{(n+1)}(\xi)}{(n+1)!}(x-c)^{n+1}$$

An error bound for the approximation error $E$ is given by

$$|E| = |f(x) - P_n(x)| \leq \frac{M}{(n+1)!}|x-c|^{n+1}$$

where

$$|f^{(n+1)}(x)| \leq M$$
on the interval containing $c$ and $x$. 

Exploration 1  Taylor Polynomials and the Sine Function

In this exploration, we will use the sine function and its Taylor polynomials to help us develop a better understanding of the approximating properties of Taylor polynomials.

1. What is the Taylor Series for $f(x) = \sin(x)$ about $c = 0$?

2. Find the following Taylor polynomials for $f(x) = \sin(x)$ about $c = 0$:
   
   $P_1(x) =$ .................................................................
   
   $P_3(x) =$ .................................................................
   
   $P_5(x) =$ .................................................................
   
   $P_7(x) =$ .................................................................
   
   $P_9(x) =$ .................................................................

To begin this exploration, we need to set up TEMATH by doing the following:

- Click **off** autoscaling, set the domain to $-\pi \leq x \leq \pi$ and set the range to $-2 \leq y \leq 2$. Note: Press **Option p** for $\pi$.
- Select **Pen...** from the **Options** menu. Click the box containing the $\times$ to the left of **Highlight Selected Plot** to remove the $\times$. Click the **OK** button. All graphs will now be plotted with a thin line. This will make it visually easier to compare the approximating polynomials.
- Enter and plot $f(x) = \sin(x)$.
- Enter and overlay the plot of the first degree Taylor polynomial $P_1(x) = x$.
- Enter and overlay the plot of the third degree Taylor polynomial $P_3(x) = x - \frac{x^3}{3!}$.

Continue to enter and overlay the plots of higher degree Taylor polynomials until you find one that “visually” matches the graph of $f(x) = \sin(x)$ on the interval $-\pi \leq x \leq \pi$.

3. As $n$ increases ($n = 1, 3, 5, \ldots$), describe how the Taylor polynomials $P_n(x)$ become better approximations for $\sin(x)$.

   ..........................................................................
   
   ..........................................................................
   
   ..........................................................................
   
   ..........................................................................
   
   ..........................................................................

Kowalczyk & Hausknecht 8/14/00
4. What is the lowest degree Taylor polynomial that “visually” matches the graph of \( f(x) = \sin(x) \) on the interval \(-\pi \leq x \leq \pi \)? ........................................................

Is this Taylor polynomial and \( \sin(x) \) exactly the same? If not, by how much do they differ? To find out,

- Click on autoscaling in the Domain & Range window.
- Enter and plot the absolute value of the error function \( |E_n(x)| = |\sin(x) - P_n(x)| \) for the Taylor polynomial given in question 4 above. Note: If, for example, \( P_n(x) \) is \( y_6(x) \) in the Work window, you can enter \( |E_n(x)| \) as \( \text{abs}(\sin(x) - y_6(x)) \).

5. a) What is the largest magnitude of the approximation error \( |E_n(x)| \) on the interval \(-\pi \leq x \leq \pi \)? ........................................................

b) Find an upper bound for the Lagrange remainder term \( |R_n(x)| \). Show all your work and give reasons for each step........................................................

........................................................

........................................................

........................................................

........................................................

c) How does the error bound in part b) compare with the actual largest error (in magnitude) found in part a)? ........................................................

........................................................

........................................................

........................................................

6. a) What is the lowest degree Taylor polynomial that approximates the sine function on \(-\pi \leq x \leq \pi \) correct to four decimal places, that is, you want \( |E_n(x)| \leq 0.5 \times 10^{-4} \)? ........................................................

b) Describe how you found this polynomial........................................................

........................................................

........................................................

........................................................
You can use TEMATH to evaluate functions given in the Work window. For example, suppose you want to find the value of the function y8(x) at \( x = 3 \), that is, you want to find \( y8(3) \). To do this,

- Select **Calculators — Expression Calculator...** from the **Work** menu.
- When the Expression Calculator opens, enter the expression \( y8(3) \) and press the **Enter** key or click the **Evaluate** button. Be sure that the flashing cursor is on the same line as \( y8(3) \) when you press the Enter key. The value of \( y8(3) \) will be written to the next line.
- To find \( y8(17) \), simply edit the expression \( y8(3) \) to read \( y8(17) \) and press the **Enter** key.

You can also enter and evaluate expressions like \( \sin(3) - y8(3) \) by doing the following:

- If you are not at the beginning of a blank line in the Expression Calculator, press the **Return** key.
- Enter the expression \( \sin(3) - y8(3) \) and press the **Enter** key. The value will be written to the next line.

### 7.

a) Using the polynomial you found in question 6, what is an approximate value for \( \sin(2) \)?

b) What is the error of this approximation?

---

### Exploration 2  
**Interval of Convergence**

The Taylor series for \( \ln(x) \) about \( x = 1 \) is

\[
\ln(x) = (x - 1) - \frac{(x - 1)^2}{2} + \frac{(x - 1)^3}{3} - \cdots + (-1)^{(n+1)} \frac{(x - 1)^n}{n} + \cdots
\]

\[
= \sum_{i=1}^{\infty} (-1)^{i+1} \frac{(x - 1)^i}{i}
\]
In this exploration, we want to use graphs to visually estimate the interval of convergence of this Taylor series. To do this, we need to do the following:

- If autoscaling is on, click it off.
- Set the domain to $-1 \leq x \leq 3$ and set the range to $-5 \leq y \leq 3$.
- Enter and plot the function $f(x) = \ln(x)$.
- Enter the _TEMATH_ expression $\sum(i, 1, 10, (-1)^{(i+1})(x-1)^i/i)$ (press _Option w_ for _\sum_) for the 10th degree Taylor polynomial for $\ln(x)$ about $x = 1$.
- Select _Overlay_ plot from the _Graph_ menu.
- Enter the _TEMATH_ expression $\sum(i, 1, 25, (-1)^{(i+1)}(x-1)^i/i)$ for the 25th degree Taylor polynomial.
- Select _Overlay_ plot from the _Graph_ menu.
- Enter and overlay the plots of the 50th, 75th, and 100th degree Taylor polynomials.

Attach to this lab report a printed copy of this final graph.

1. Using these graphs, visually estimate the interval of convergence of the Taylor series for $\ln(x)$ about $x = 1$. 

   Give reasons why you selected this interval. 

2. Using analytical tools, find the exact interval of convergence of the Taylor series for $\ln(x)$ about $x = 1$. Show all your work.
3. Find the Taylor series for \( f(x) = \frac{1}{x + 2} \) about \( x = 0 \) ..............................................
                                                                                     
4. Using the graphical technique described above, visually estimate the interval of convergence of the Taylor series for \( f(x) = \frac{1}{x + 2} \) about \( x = 0 \) ..............................................
                                                                                     
Attach to this lab report a printed copy of the final graph used to obtain the answer for question 4.

5. Using one of the standard tests for convergence, find the exact interval of convergence of the Taylor series for \( f(x) = \frac{1}{x + 2} \) about \( x = 0 \). Show all your work......................
                                                                                     
6. In your own words, give a definition for “Interval of Convergence”......................
                                                                                     
Kowalczyk & Hausknecht 8/14/00