Modern calculus courses are incorporating many more applications and special functions into their content. Using and applying these special functions is oftentimes a complete mystery to our students. To help them better explore real-life applications and the properties of special functions, we have developed a special set of visual tools that we use with our students. In this paper, we present a collection of examples that demonstrate the use of these visual tools.

Defining a Special Function

In the first week of our calculus courses when we first introduce the concept of a function to our students, we talk about input, output, and the rule describing the function. TEMATH’s Rectangular Tracker tool is used to help students visualize the concept of inputting a value of $x$, outputting a value of $y$, and using a graph to represent this process (see Figure 1). A few weeks later, we introduce the concept of the derivative as a function. From a purely algebraic point of view, this is quite bewildering to our students. We use TEMATH’s Dynamic Tangent tool to help students visualize the derivative as a function where the input is a value of $x$ and the output is the slope of the tangent line to the curve. As the tangent dynamically moves along the curve of the function $f(x)$, the value of the slope of the tangent line is plotted creating the graph of the derivative function $f'(x)$ (see Figure 2). This visual demonstration helps students develop an intuition about the derivative as a function, and more importantly, as a function that measures the instantaneous rate of change of $f(x)$.

Our calculus students readily accept functions like $f(x) = x + \sqrt{x}$ and $g(x) = 3\sin(2x)$, but when we use the Fundamental Theorem of Calculus to start constructing antideriva-

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tives like $\text{Si}(x) = \int_{0}^{x} \frac{\sin(t)}{t} dt$, and then actually call $\text{Si}(x)$ a function that has applications to optics, our students look on with bewilderment. This is when we use technology and its powerful visual tools to help our students visualize the concept of this type of function definition. We use TEMATH’s Dynamic Integrator tool to visually demonstrate the definition of $\text{Si}(x)$ and to explore its properties. This tool dynamically shades in the “area so far” and plots the value of that area, thus, generating the plot of $\text{Si}(x)$ (see Figure 3).

![Figure 3 The Sine-Integral](image1)
![Figure 4 Nautilus Shell Data Sampling](image2)

**Modeling the Spiral of the Chambered Nautilus Shell**

Nature presents many excellent opportunities for using mathematics to describe the structure of its various life forms. For example, polar coordinates can be used to model the spiral structures found in many shells, in particular, the Chambered Nautilus. We used a digital camera to take a picture of a Chambered Nautilus shell, imported the image into a computer, and copied it into TEMATH’s Polar Plot Mode. Using TEMATH’s Point tool, we sampled points along the shell’s spiral at intervals of $\pi/4$ radians (see Figure 4). Next, we created a table of values for these points, found the least squares

![Figure 5 Least Squares Exponential Fit](image3)
![Figure 6 Nautilus Shell Spiral Fit](image4)

exponential fit \( r = 0.0121716424958 e^{0.169515095317 t} \) using rectangular coordinates (see Figure 5), and overlaid the polar plot of the fit on top of the image of the shell (see Figure 6). The fit is excellent! Real-life applications truly make our students appreciate the modeling potential of mathematical functions.

**Modeling a Hanging Chain**

In many traditional elementary differential equation texts, Bessel’s differential equation,

\[ t^2 y'' + ty' + (1 - v^2) y = 0 \]

where \( v \geq 0 \) is a parameter, is used to introduce or illustrate power series techniques for solving differential equations around regular singular points. The geometric properties of its solutions are rarely discussed and its applications are only mentioned in passing. Thus, a student’s first exposure to Bessel’s equation is almost always from the algebraic point of view (see [5]). To motivate student interest in Bessel’s equation, we use TEMATH’s differential equation solver to study the equation. For example, if \( v = 0 \), then we can visually verify that

\[
J_0(t) = \sum_{k=0}^{\infty} \frac{(-1)^k}{(k!)^2} \left( \frac{x}{2} \right)^{2k} \quad \text{and} \quad Y_0(t) = \frac{\pi}{2} J_0(t)(\gamma + \ln(x/2)) - \frac{\pi}{2} \sum_{k=0}^{\infty} \frac{(-1)^k}{(k!)^2} \phi(k) \left( \frac{x}{2} \right)^{2k}
\]

(where \( \gamma = 0.577215664902\ldots \) is Euler’s constant and \( \phi(k) = 1 + \frac{1}{2} + \cdots + \frac{1}{k} \)) are independent solutions (see Figure 7). It is easy to see that they are independent because

\[
J_0(0) = 1 \quad \text{and} \quad \lim_{t \to 0} Y_0(t) = -\infty.
\]

Small oscillations of a hanging chain can be modeled by the partial differential equation

\[
\frac{\partial^2 x}{\partial t^2} = g \frac{\partial}{\partial y} \left( y \frac{\partial x}{\partial y} \right)
\]

where \( g \) is acceleration due to gravity (see Figure 8). Because the partial derivatives on the left are with respect to \( t \) alone and

\footnote{This article will appear in "Proceedings of the Twelfth Annual International Conference on Technology in Collegiate Mathematics", Addison-Wesley, 2001.}
the partial derivatives on the right are with respect to $y$ alone, sums of functions of the form $x(y,t) = f(y)\cos(gt + e)$ are solutions where $f(y)$ is a differentiable function such that $f(L) = 0$ and $g, e$ are constants. Making the substitutions, $x = f(y)\cos(gt + e)$ and 
$y = gu^2/(2w)^2$ , reduces the model to Bessel’s equation 
$u''F + uu'F' + F = 0$ where 
$F(u) = (f \circ y)(u)$. Since $Y_0(0)$ is undefined, the solution of the reduced equation must be 
$F(u) = cJ_0(u)$. Transforming back, we find that $f(y) = cJ_0(r_1yL)$ where $r_1$ is a root of $J_0$. Consequently, 
$x(y,t) = \sum_k c_k J_0(r_kyL)\cos(\sqrt{g/L} \frac{r_k}{2} t - e_k)$ are solutions of the original model where $r_k$ is the $k$-th root of $J_0(t)$, $L$ is the length of the chain, and $c_k, e_k$ are constants (see [1], [4], [5] for details and history). A sequence of plots of the normal mode solution 
$x(y,t) = J_0(r_1y)\cos(\sqrt{2L} \frac{r_1}{2} t)$ is shown in Figure 9. A sequence of plots for the multi-mode solution 
$x(y,t) = \frac{1}{2} \sum_{k=1}^{2} J_0(r_ky)\cos(\sqrt{2L} \frac{r_k}{2} t)$, is shown in Figure 10.

\begin{figure}[h]
\centering
\begin{subfigure}{0.45\textwidth}
\centering
\includegraphics[width=\linewidth]{normal_mode_solution.png}
\caption{A Normal Mode Solution}
\end{subfigure}
\hfill
\begin{subfigure}{0.45\textwidth}
\centering
\includegraphics[width=\linewidth]{multi_mode_solution.png}
\caption{A Multi-Mode Solution}
\end{subfigure}
\end{figure}

Using TEMATH 2.0, we can save the overlaid plots of the hanging cable shown in Figure 9 (or Figure 10) as a sequence of separate picture files which then can be converted into an animation using a variety of shareware software such as GraphicConverter™. To motivate class discussion, you can set up a hanging chain in class and compare its motion to the animation generated by the Bessel function model.

**Visualizing Newton’s Method**

Newton’s method converges rapidly to a root if the initial guess is nearby or the function near the root is “well-behaved.” To help our students visualize the convergence and the pitfalls of Newton’s Method, we use TEMATH’s Root Finder Tool. We start with a straightforward example like $f(x) = x^2 - 2$. After an initial guess is entered into

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TEMATH, each click of the “Step” button performs one iteration, draws the tangent to the curve, and displays the next iterate. Students watch the tangent line process converge rapidly (within six iterations) to the root (see Figure 11). Next we present the students with a more challenging example, \( f(x) = x^3 - x - 3 \). Starting with an initial guess of \( x = 0 \), Newton’s iteration process continuously cycles between the values 0, –3, –1.96, and –1.15 (see Figure 12). As another example exhibiting a different behavior, we use the arctan function. We start with an initial guess of \( x_0 = 1.39174 \). At the beginning, the iterates appear to cycle back and forth between two values, but then after about ten iterations, the iterates converge rapidly to the root \( x = 0 \). If we slightly increase the value of the initial guess to \( x_0 = 1.39175 \), the iterates again appear to cycle back and forth between two values, but then after about ten iterations, the iterates diverge towards both \( \pm \infty \). These visualizations help our students develop a real understanding of the convergence/divergence properties of Newton’s method.

![Figure 11 Newton’s Method](image1.png)

![Figure 12 Cycling with Newton’s Method](image2.png)

**Bibliography**


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