Dynamic price competition with discrete customer choices

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Abstract

For many years, dynamic pricing has proven to be an effective tool to increase revenue in the airline and other service industries. Most studies, however, focused on monopolistic models and ignored the fact that nowadays consumers can easily compare prices on the Internet. In this paper, we develop a game-theoretic model to describe real-time dynamic price competition between firms that sell substitutable products. By assuming the real-time inventory levels of all firms are public information, we show the existence of Nash equilibrium. We then discuss how a firm can adapt if it knows only the initial – but not the real-time – inventory levels of its competitors. We compare a firm’s expected revenue under different information structures through numerical experiments.

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1. Introduction and overview

Dynamic pricing is a business strategy that adjusts the product price in real-time in order to maximize profit. Dynamic pricing has been widely used in the travel industries, such as airlines, rental cars, hotels, cruise lines, and railroad services; for example, see Bitran and Mondschein (1995), Gallego and van Ryzin (1997), Ladany and Arbel (1991), You (1999), and the references therein. The reason these travel industries are particularly suitable for dynamic pricing is that the inventory cannot be replenished and unsold products have little salvage value. A firm has incentive to increase the price if the sale goes well initially in order to reserve products for potential later customers who may be willing to pay more. On the other hand, if the sale does not go well in the beginning, a firm may want to lower the price to induce sales because unsold products have little salvage value. Interested readers can refer to Elmaghraby and Keskinocak (2003), McGill and van Ryzin (1999), or Weatherford and Bodily (1992) for a general survey on dynamic pricing and its role in revenue management.

The earliest work concerning dynamic pricing appears to be that of Kincaid and Darling (1963). Since the deregulation of the US airline industry in the 1970s, the research on dynamic pricing continued to grow to provide tools for the industries. The majority of research published in the open literature, however, assumes the firm enjoys monopolistic power. An important assumption of such a monopolistic model is that the distribution of the random demand depends solely on the price set by the monopolist. Although this assumption is quite realistic in the past when it is difficult for a customer to compare prices, it can be problematic nowadays because of the Internet. With a few clicks on the keyboard, a customer can easily extract the real-time price of an airline ticket. Although there are loyal customers (memberships, personal preferences, or other reasons) who will always choose a particular airline, many customers have schedules flexible enough to take either itinerary, and will take into account the price when purchasing the tickets. Consequently, the demand for a
product depends not only on its price, but also on the price of a similar product from a different supplier. This observation motivates our work in this paper.

Our work makes three significant contributions to existing literature on dynamic pricing: (1) when real-time inventory levels are public information, we show the existence of Nash equilibrium in a multistage game; (2) we show that the price monotonicity found in most monopolistic models need not hold true when there is competition; and (3) we develop a heuristic policy that requires only the initial – but not the real-time – inventory levels of a firm’s competitors, and numerically evaluate its performance under different information structures.

### 1.1. Related research

Monopolistic dynamic pricing models have been well studied in the literature, such as those studied by Chatwin (2000), Feng and Xiao (2000), Gallego and van Ryzin (1994, 1997), Lin (2004) and Zhao and Zheng (2000). In the stream of this work, a firm that enjoys monopolistic power uses its price as a tool to induce demand, with the objective to maximize the total expected revenue when the sale ends. Because the optimal policy is often difficult to find analytically, in many cases the authors first establish structural properties of the optimal policy, and then develop an algorithm to compute or to approximate it. In other cases, the authors develop efficient heuristic policies.

There is limited research in revenue management of perishable products that concerns competition between firms. Dudew (1992) studies a duopoly model where two firms offer identical products so a customer always chooses the one with a lower price. Li and Oum (1997) and Netessine and Shumsky (2005), respectively, examine the competition between two airlines, each of which has two fare classes. The prices of two fare classes are identical for the two airlines, and each airline has to decide how many seats to reserve for the high fare class. However, the competition is static because the seat allocation decision is made before the tickets go on sale, and cannot be changed later on. Perakis and Sood (2004) provides a multiple-period pricing model for perishable products in a competitive market. The demand faced by the sellers is a deterministic function of the prices set by all sellers. In their model, at the beginning of the sales horizon the sellers solve a one-time optimization problem to set the price path, rather than setting the price at the beginning of each time period according to the real-time inventory levels. Our game-theoretic model differs from these studies because our model allows firms to dynamically set product prices in real-time.

There is a growing interest in recent years on competition between retailers in the context of supply chain management. For competition models where a retailer chooses a capacity (no inventory replenishment) and sets a price (no dynamic pricing), see Benoit and Krishna (1987), Herk (1993), and the references therein. In some other works, retailers can replenish their inventory. Common criteria for retailers are to maximize expected profit in a finite-time setting, or to maximize the long-run average profit on an infinite-time horizon. For recent examples, see Cachon and Zipkin (1999), Hopp and Xu (2003), Lippman and McCardle (1997), Mahajan and van Ryzin (2001), Netessine and Rudi (2003), and the references therein. Papers that incorporate the price as a decision variable and allow inventory replenishment includes, for example, Bernstein and Federgruen (2003, 2004) and Kirman and Sobel (1974). However, conditional on the total realized demand, the demand for each firm is often formulated as a deterministic function of prices. To the best of our knowledge, there is no published papers that study price competition with discrete customer choices. Our work in this paper fills this research gap.

### 1.2. Overview and outline

In this paper, we develop a game-theoretic model to describe the dynamic price competition between firms who offer substitutable products. Each firm offers one product and starts with a given number of items in the beginning of sale. We use a discrete-time model with the length of a time period short enough so that in each time period at most one customer can arrive. As the sale goes on, a firm can dynamically change its product price in each time period. When a customer arrives, he first compares these substitutable products and their prices, and then either buys one item of his most preferable product, or leaves empty-handed. Specifically, we use the multinomial logic choice model – which is commonly used in marketing science (for example, see Franses and Paap (2001) and Lilien et al. (1992)) – to describe a customer’s discrete choice. A major distinction of our model from existing dynamic pricing models is that the probability a customer purchases a product not only depends on its price, but also on prices of other substitutable products. The goal of each firm is to maximize the expected total revenue when the sale ends.

Because there are multiple firms in the game, each firm’s optimal policy depends on what other firms do. By assuming each firm knows the real-time inventory levels of all other firms, we characterize the price and the expected revenue in Nash equilibrium. What is somehow surprising is that the price in Nash equilibrium does not exhibit some common structural properties one would expect in a monopolistic dynamic pricing model. For example, a monopolistic firm should set a higher price when the demand becomes larger or when the supply becomes smaller. In an oligopoly, however, the preceding needs not be true in general. This observation has significant managerial implication.
Although it is theoretically appealing to assume the real-time inventory levels of all firms are public information and to establish a Nash equilibrium based on this assumption, in practice, however, keeping track of the inventory levels of the other firms can be expensive, or sometimes impossible. For example, it is relatively easy to know the capacity of the aircraft used by the other airline, but difficult to know the number of seats still available. When the real-time inventory levels of the other firms are not available, we propose a heuristic policy based on our findings from the full information model. We use numerical examples to evaluate the heuristic policy under different information structures.

The rest of the paper is organized as follows. Section 2 describes our game-theoretic model. Section 3 concerns the case of complete information such that the real-time inventory levels of all firms are public information. We show the existence of Nash equilibrium and present a counterexample to price monotonicity often found in monopolistic models. Section 4 discusses the case of incomplete information when a firm knows only the initial – but not the real-time – inventory levels of the other firms. We develop a heuristic policy and numerically compare the expected revenue under different information structures. Finally, Section 5 concludes the paper and points out future research directions.

2. Model and preliminaries

Consider $n$ firms that sell substitutable products over $T$ time periods, where we count the time period in the reverse chronological order so that period 1 represents the last period. In period $T$, firm $j$ starts with $c_j$ items of its product in inventory, $j = 1, \ldots, n$. Inventory cannot be replenished during the sale and unsold items at the end of period 1 have no salvage value. Customers arrive from period 1 to period 1 according to a Bernoulli process such that in each period the probability a customer arrives is $\lambda$ and the probability that no customer arrives is $1 - \lambda$. This assumption of customer arrival process can be used to approximate a nonhomogeneous Poisson process (with proper time scale), and was commonly used in revenue management literature; for example, see Lautenbacher and Stidham (1999), Subramanian et al. (1999) and You (1999).

At the beginning of each time period, each firm sets its product price. A customer compares the prices of all products and will leave empty-handed with probability $\frac{1}{\sum_{j=1}^{n} e^{z_{ij} - \frac{p_j}{c_j}}}$.

Then considers whether to buy one unit of his most preferable product or not to buy at all. In this paper, we use the multinomial logit (MNL) model to describe the customer’s discrete choice. The MNL model is widely used in marketing science literature, and it assumes that each customer acts independently to maximize its own utility; see Lilien et al. (1992) or Franses and Paap (2001) for more discussions on the MNL model. Specifically, for each customer, the utility for purchasing one product from firm $i$ at price $p_i$ is equal to

$$U_i = x_i - \beta p_i + Z_i, \quad i = 1, \ldots, n,$$

and the utility of no purchase is

$$U_0 = Z_0.$$

The parameter $x_i$ models the quality, brand image, and the popularity of firm $i$’s product, and $\beta$ represents the price response coefficient. The random variables $Z_i, i = 0, \ldots, n$, describe the idiosyncratic preference of each customer, and are independent and identically distributed Gumbel random variables with the following distribution function:

$$P(Z_i \leq z) = \exp(-(e^{z - \frac{\mu}{\eta}})), \quad z \in (-\infty, \infty),$$

where $\mu$ is a shift parameter and $\eta$ is a scale parameter. When facing a price vector $p = (p_1, \ldots, p_n)$, where $p_i$ denotes the price set by firm $i$, a customer will maximize its utility. In other words, a customer will buy one item from firm $i$ with probability (see Ben-Akiva and Lerman (1985) for a derivation)

$$q_i(p) = P(U_i = \max_{j=0, \ldots, n} U_j) = \frac{e^{x_i - \beta p_i}}{1 + \sum_{j=1}^{n} e^{x_j - \beta p_j}}, \quad i = 1, \ldots, n,$$

and will leave empty-handed with probability

$$q_0(p) = \frac{1}{1 + \sum_{j=1}^{n} e^{x_j - \beta p_j}}.$$  

When calculating $q_i(p)$, for $i = 0, \ldots, n$, we have let $\mu = 0$ and $\eta = 1$ without loss of generality, because these parameters can be absorbed into the constants $x_i, i = 1, \ldots, n$, and $\beta$.

From Eq. (1), we see that $q_i(p)$ – the probability firm $i$ sells one item – is monotonically decreasing in its price $p_i$, and that

$$\lim_{p_i \to \infty} q_i(p) = \lim_{p_i \to \infty} p_i q_i(p) = 0, \quad i = 1, \ldots, n.$$
Therefore, a firm can set the price to infinity if it does not want to offer its product to a particular customer, in which case the expected revenue is equal to 0. In addition, if a firm sells out its product before the sale ends, then its product price – from the customers’ and the other firms’ standpoints – becomes infinity for the rest of the sales horizon. The objective for each firm is to dynamically set its product price in order to maximize the expected total revenue when it sells out its entire stock or when the sale ends.

3. The case of complete information

This section concerns the situation in which each firm can track the real-time inventory levels of all other firms. In the beginning of each time period, a firm first finds out the respective number of items every other firm still has, and then chooses its product price. In Section 3.1, we prove the existence of Nash equilibrium of this multistage game with complete information. In Section 3.2, we discuss the price monotonicity that is commonly found in a monopolistic model.

3.1. Existence of Nash equilibrium

To prove the existence of Nash equilibrium, we first consider the single-period game when $T = 1$. Suppose $c_i \geq 1$ for $i = 1, \ldots, n$. Because there is only one-time period left, the common objective for all firms is to maximize the expected revenue from the arriving customer. Let $p_i$ denote the price set by firm $i$, for $i = 1, \ldots, n$, and $p = (p_1, \ldots, p_n)$ the joint price vector. The payoff function (the immediate expected revenue from the arriving customer) for firm $i$ is

$$\pi_i(p) \equiv p_i \cdot q_i(p) = \frac{p e^{x_i - l p_i}}{1 + \sum_{j=1}^n e^{x_j - l p_j}}, \quad i = 1, \ldots, n.$$  \hspace{1cm} (3)

Bernstein and Federgruen (2004) showed that this single-period game has a unique Nash equilibrium, in which each firm’s price is the best response to the prices posted by the others, so that no firm has incentive to change its price. In the rest of this paper, we will refer a firm’s price in the unique Nash equilibrium of the single-period model as the myopic price, because each firm maximizes the immediate expected revenue and does not take into account how the potential sale might affect its inventory and therefore its revenue in the future. These myopic prices would indeed describe the Nash equilibrium if a customer arrives in the last period, or if each firm has unlimited inventory.

To define a Nash equilibrium when there are $t \geq 2$ time periods remaining on the sales horizon, we use mathematical induction and assume that a Nash equilibrium exists for periods 1, 2, \ldots, $t - 1$. Denote the current inventory vector by $s = (s_1, \ldots, s_n)$ with the interpretation that firm $i$ still has $s_i$ items in inventory, $i = 1, \ldots, n$. Moreover, let $V_i(s, t - 1)$ denote firm $i$’s expected total revenue from period $t - 1$ to period 1 if all firms use equilibrium strategies from period $t - 1$ to period 1. The boundary condition requires that $V_i(s, 0) = 0$ for all $i$ and all $s$, and $V_i(s, t) = 0$ if $s_i = 0$ and $t = 1, 2, \ldots$. Consider the beginning of period $t$. If no customer arrives in period $t$, then the firms simply carry over their respective inventories to the beginning of period $t - 1$. On the other hand, if a customer arrives in period $t$, then the game between the firms is such that firm $i$ chooses $p_i$ to maximize its payoff function $\Pi_i(p, s, t)$, $i = 1, \ldots, n$, where

$$\Pi_i(p, s, t) \equiv q_i(p)(p_i + V_i(s - e_i, t - 1)) + \sum_{j=1, j \neq i}^n q_j(p)V_i(s - e_j, t - 1) + q_0(p)V_i(s, t - 1)$$

$$= p_i + V_i(s - e_i, t - 1) + \sum_{j=0, j \neq i}^n q_j(p)(V_i(s - e_j, t - 1) - V_i(s - e_i, t - 1) - p_i), \hspace{1cm} (4)$$

where $e_i$ represents a $1 \times n$ vector whose $i$th entry is equal to 1 and all others equal to 0, and $e_0 = (0, 0, \ldots, 0)$ is the zero vector.

If there exists a Nash equilibrium – denoted by $p^*(s, t)$ – in the game whose payoff functions are specified by Eq. (4), then we can define the equilibrium expected revenue for firm $i$ from period $t$ to period 1 by

$$V_i(s, t) = \lambda \Pi_i(p^*(s, t), s, t) + (1 - \lambda)V_i(s, t - 1),$$

because a customer shows up in each period with probability $\lambda$. To validate this approach, we first present two lemmas, based on which we then give the main theorem that guarantees the existence of a Nash equilibrium at the beginning of each period.

We say a function $f(x)$ is strongly quasi-concave in $x$ if there exists an $\hat{x}$ such that $f(x)$ increases in $x$ for $x < \hat{x}$, and decreases in $x$ for $x > \hat{x}$.

**Lemma 3.1.** The payoff function $\Pi_i(p, s, t)$ is strongly quasi-concave in $p_i$.

**Proof.** Taking the first derivative of $\Pi_i(p, s, t)$ with respect to $p_i$ yields
The best response function is uniformly bounded. That is, 
\[
\frac{\partial \Pi_i(p, s, t)}{\partial p_i} = q_i(p) \left(1 - \beta \sum_{j=0, j \neq i}^{n} q_j(p)(p_i + V_i(s - e_i, t - 1) - V_i(s - e_j, t - 1))\right),
\]  
(5)

Let \( h_i(p) \equiv 1 - \beta \sum_{j=0, j \neq i}^{n} q_j(p)(p_i + V_i(s - e_i, t - 1) - V_i(s - e_j, t - 1)) \). Fix \( p_1, \ldots, p_{i-1}, p_{i+1}, \ldots, p_n \), and consider the following two cases:

1. The function \( h_i(p) \neq 0 \) for all \( p_i \geq 0 \); because \( h(p_i) \) is continuous and \( \lim_{p_i \to \infty} h(p_i) = -\infty \), it follows that \( h(p_i) < 0 \) for \( p_i \geq 0 \). Hence, \( \Pi_i(p, s, t) \) strictly decreases in \( p_i \) for \( p_i \geq 0 \), and therefore is strongly quasi-concave.

2. There exists \( \hat{p}_i \) such that \( h_i(p_1, \ldots, p_{i-1}, \hat{p}_i, p_{i+1}, \ldots, p_n) = 0 \). The second derivative of \( \Pi_i(p, s, t) \) with respect to \( p_i \), evaluated at \( \hat{p}_i \), is

\[
\frac{\partial^2 \Pi_i(p, s, t)}{\partial p_i^2} \bigg|_{p_i=\hat{p}_i} = \left(\frac{\partial q_i(p)}{\partial p_i} h_i(p) \right) \bigg|_{p_i=\hat{p}_i} + \left( q_i(p) \frac{\partial h_i(p)}{\partial p_i} \right) \bigg|_{p_i=\hat{p}_i} = 0 + q_i(p)(-\beta) \sum_{j=0, j \neq i}^{n} q_j(p) + \sum_{j=0, j \neq i}^{n} (\beta q_i(p) q_j(p)(p_i + V_i(s - e_i, t - 1) - V_i(s - e_j, t - 1))) \bigg|_{p_i=\hat{p}_i} = -\beta q_i(p) \sum_{j=0, j \neq i}^{n} q_j(p),
\]

where the third equality follows from \( h_i(p) \big|_{p_i=\hat{p}_i} = 0 \). The preceding shows that \( \hat{p}_i \) is a local maximum of \( \Pi_i(p, s, t) \) and that there does not exist an interior minimum for \( p_i \in [0, \infty) \). It then follows that \( \hat{p}_i \) is unique because otherwise there must exist an interior minimum for \( \Pi_i(p, s, t) \). Consequently, the function \( \Pi_i(p, s, t) \) increases for \( p_i \in [0, \hat{p}_i) \) and decreases for \( p_i \in (\hat{p}_i, \infty) \), and therefore is strongly quasi-concave.

The lemma follows because \( \Pi_i(p, s, t) \) is strongly quasi-concave in \( p_i \) in both cases. \( \square \)

From the preceding lemma, we can define the best response function when a customer arrives in period \( t \) and firm \( i \) still has \( s_i \) items in inventory, \( i = 1, \ldots, n \), as follows:

\[
\Phi_i(p_{-i}, s, t) \equiv \arg \max_{p_i} \Pi_i(p, s, t).
\]  
(6)

We next show that this best response function is uniformly bounded.

**Lemma 3.2.** The best response function is uniformly bounded. That is, \( \sup_{p_i} \Phi_i(p_{-i}, s, t) < \infty \).

**Proof.** In state \((s, t - 1)\), denote \( b_{ij} = -V_i(s - e_i, t - 1) + V_i(s - e_j, t - 1) \), and let \( b_i = \max_j b_{ij} \), which is some constant. Rewrite Eq. (5) as follows:

\[
\frac{\partial \Pi_i(p, s, t)}{\partial p_i} = q_i(p) \left(1 - \beta \sum_{j=0, j \neq i}^{n} q_j(p)(p_i - b_{ij})\right) = q_i(p)(1 - \beta(1 - q_i(p))(p_i - b_i)).
\]  
(7)

Because

\[
q_i(p) = \frac{e^{\alpha_i - \beta p_i}}{1 + \sum_{j=1}^{n} e^{\alpha_j - \beta p_j}} \leq \frac{e^{\alpha_i - \beta p_i}}{1 + e^{\alpha_i - \beta p_i}} \leq \frac{e^{\alpha_i}}{1 + e^{\alpha_i}},
\]  
(8)

Eq. (7) becomes negative if

\[
p_i > b_i + \frac{1 + e^{\alpha_i}}{\beta}.
\]

In other words, \( \Pi_i(p, s, t) \) decreases in \( p_i \) for \( p_i > b_i + (1 + e^{\alpha_i})/\beta \) regardless what prices the other firms choose. Therefore, firm \( i \) does not need to consider any price that is greater than \( b_i + (1 + e^{\alpha_i})/\beta \). Consequently, we can conclude that \( \sup_{p_i} \Phi_i(p_{-i}, s, t) \leq b_i + (1 + e^{\alpha_i})/\beta \), which completes the proof. \( \square \)
Theorem 3.1. There exists at least one pure-strategy Nash equilibrium in each period in the multiple-period game.

Proof. The existence of a pure-strategy Nash equilibrium follows from Theorem 2.1 in Vives (1999) – originally attributed to Debreu (1952) – which states that a Nash equilibrium exists if the strategy sets are nonempty convex and compact, and the payoff to firm $i$ is continuous in the actions of all firms and quasi-concave in its own action. In our multiple-period model, although each firm can choose a price from $[0, \infty )$, which is not compact, Lemma 3.2 allows us to construct an equivalent game by restricting firm $i$ to choose a price from a nonempty convex and compact set. In addition, as seen in Eq. (4), the payoff function $\Pi_i (p, s, t)$ is continuous in $p_j$, $j = 1, \ldots, n$. Finally, according to Lemma 3.1, the payoff function $\Pi_i (p, s, t)$ is quasi-concave in $p_i$. Consequently, a pure-strategy Nash equilibrium exists. □

The proof of the uniqueness of Nash equilibrium is much more difficult because we cannot quantify the terms $V_i (\cdot, \cdot)$ in a firm’s payoff function, and standard methods – such as diagonal dominance condition – turn out to be fruitless. Below we offer a plausible way to choose one Nash equilibrium if there are multiple Nash equilibria. In numerical experiments, however, we have not encountered any instance with multiple Nash equilibria.

Let $p^* (s, t)$ denote one Nash equilibrium with inventory vector $s$ and $t$ time periods to go. At this equilibrium, the payoff function of firm $i$ can be computed by Eq. (4) as follows:

$$
\Pi_i (p^* (s, t), s, t) = p_i^* (s, t) + V_i (s - e_i, t - 1) + \sum_{j=0, j \neq i}^{n} q_j (p^* (s, t)) (V_i (s - e_j, t - 1) - V_i (s - e_j, t - 1) - p_j^* (s, t))
$$

where the second equality follows because $p^* (s, t)$ must satisfy Eq. (5). In other words, a firm’s payoff function increases linearly in its equilibrium price if there are more than one Nash equilibrium. Naturally, if there are multiple equilibria, each firm prefers to play its highest equilibrium price. To ensure our notations are consistent, we use $p_i^* (s, t)$ to denote the highest Nash equilibrium price for firm 1 in period $t$ if the inventory vector is $s$, and use $p_i^* (s, t)$ to denote the Nash equilibrium price for each firm $i$, $i \neq 1$, when firm 1 sets its price to $p_1^* (s, t)$. In addition, the value function $V_i (\cdot, \cdot)$ for firm $i$, $i = 1, \ldots, n$, are defined according to this Nash equilibrium.

### 3.2. On price monotonicity

When a firm dynamically sets its product price in a monopolistic setting, a rule of thumb is that the price should be set higher if, loosely speaking, the demand becomes larger or if the supply becomes smaller. Specifically, with the same number of items in inventory, the firm should set a higher price if there is more time remaining. Somewhat surprisingly, the price monotonicity in remaining time for sale does not hold true in general when there is competition. Consider a two-firm example in which $x_1 = 5$, $x_2 = 4$, $\beta = 0.1$, $\lambda = 0.1$, and initial inventory levels $c_1 = 20$, $c_2 = 10$. Table 1 reports the prices in Nash equilibrium for both firms with different values of $T$. As seen in Table 1, the price in Nash equilibrium for each firm decreases in $T$ when $T$ is relatively small, and then increases in $T$ when $T$ becomes large.

To understand this interesting phenomenon intuitively, first compute the myopic prices in the single-period game discussed in Eq. (3). In this numerical example, the myopic prices for the two firms are $p_1^i = 23.02$ and $p_2^i = 16.57$, respectively; the customer choice probabilities with respect to these myopic prices are $q_1^i = 0.5655$, $q_2^i = 0.3964$, and $q_0^i = 0.0380$.

As seen in Table 1, when $T = 20$, the total number of customers follows a binomial distribution with expected value equal to $\lambda T = 2$, and standard deviation approximately equal to 1.34. Because almost certainly the number of customers will not exceed 10, both firms have abundant inventory so that the myopic prices prevail. In this case, the number of customers dictates the number of items each firm can sell, and no matter how low the two firms reduce their prices, neither firm

<table>
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<tr>
<th>$T$</th>
<th>$p_1^i (20, 10, T)$</th>
<th>$p_2^i (20, 10, T)$</th>
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<tbody>
<tr>
<td>20</td>
<td>23.02</td>
<td>16.57</td>
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<td>40</td>
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<td>36.39</td>
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</tr>
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has a chance to use its entire inventory to generate revenue. As \( T \) increases, the possibility that the number of customers will come close to firm 2’s initial inventory level becomes significant, so firm 2 has incentive to lower its price to induce more sales. From firm 1’s standpoint, it also has incentive to lower its price not only to remain competitive, but also to help drive firm 2’s price further down—which helps firm 2 sell out more quickly so firm 1 may enjoy monopoly toward the end. Consequently, both firms benefit from lowering their prices. As \( T \) continues to increase, it becomes more and more likely that either firm will sell out its entire inventory before the sale ends if it uses the myopic (or even lower) price. Therefore, both firms should increase their prices to take advantage of the increasing number of customers.

4. The case of incomplete information

In Section 3, we discuss a multiple-period game with complete information in the sense that each firm keeps track of the real-time inventory levels of all other firms. Although the assumption of complete information and the conclusion of the existence of a Nash equilibrium are theoretically appealing, in practice, however, it may be expensive, or sometimes even impossible, for a firm to keep track of real-time inventory levels of the other firms. In this section, we consider a different information structure, in which a firm knows the initial inventory levels of the other firms, but cannot keep track of their real-time sales records. In Section 4.1, we develop a heuristic policy under this information structure. In Section 4.2, we conduct numerical experiments with a two-firm model. We evaluate this heuristic policy in different scenarios depending on the information available to the other firm and its intensity.

4.1. The interpolation heuristic

In this section, we consider the dynamic pricing problem from the standpoint of firm 1, and develop a heuristic policy that requires only the initial—but not the real-time—inventory levels of the other firms. The assumption of knowing the initial inventory levels of the other firms is reasonable; for example, the capacity of the aircraft used by another airline is usually public information.

When developing a heuristic policy, we are most interested in a policy that works well for a real-world problem. Recall from Section 3.1 that the myopic prices constitute the Nash equilibrium if each firm has more items in inventory than the maximum possible number of customers; that is, \( T \leq \min_i c_i \). In this extreme and unrealistic case, in the Nash equilibrium each firm uses a fixed-price policy (the myopic price), and dynamic pricing is not at all necessary. The effect of dynamic pricing becomes more prominent if there are more customers, or if the firms have fewer items. In a real-world problem, it is usually the case that the number of items is much smaller than the number of potential customers who inquiere product prices. For example, based on one author’s experience at United Airline, a flight’s price is usually exposed to thousands of people, and only a small portion of those people end up buying a ticket. With these observations, in our mathematical framework we are particularly interested in a dynamic pricing heuristic that works well under the condition when \( \lambda T \gg \sum_{i=1}^n c_i \) – the expected number of potential customers is much larger than the total number of items from all firms.

The rationale of our heuristic policy stems from two observations of the equilibrium prices in the complete information model discussed in Section 3. First, when \( \lambda T \gg \sum_{i=1}^n s_i \), the equilibrium price vector \( \mathbf{p}^*(s, t) = (p_1^*(s, t), \ldots, p_n^*(s, t)) \) has the property that \( \lambda q_i(\mathbf{p}(s, t)) \) – the expected number of items firm \( i \) can sell with the current equilibrium price vector – is roughly equal to \( s_i \) for \( i = 1, \ldots, n \). Second, when \( \lambda T \gg \sum_{i=1}^n s_i \), firm \( i \)'s equilibrium price \( p_i^*(s, t) \) is not sensitive to small changes in \( s_j \), \( j \neq i \). The first observation suggests that each firm intends to sell its items evenly spread out across time, and the second observation suggests that if a firm knows only roughly the inventory levels of the other firms, it can approximate the equilibrium price to a good extent.

Based on these observations, we propose a heuristic policy that interpolates the equilibrium prices. Specifically, with this heuristic policy a firm assumes that the other firms sell their products at a constant pace, and calculates its own price by interpolating the prices in Nash equilibrium. Because of the aforementioned two observations, the interpolated price should be close to the equilibrium price in the complete information model.

We illustrate in details this heuristic policy from the standpoint of firm 1 with a two-firm example as follows:

1. With \( T \) time periods to go, record firm \( i \)'s initial inventory level and denote it by \( c_i \), \( i = 1, 2 \). Set the price equal to the equilibrium price \( p_i^*(c_1, c_2, T) \) defined in Section 3.1.

2. At the beginning of period \( t \), \( t = T - 1, T - 2, \ldots, 1 \), observe the current inventory \( s_1 \), and compute \( a_2 = c_2 t / T \) and \( d_2 = \lfloor a_2 \rfloor \).

3. Set the price – which depends only on \( s_1 \), \( c_2 \), \( t \), and \( T \) – equal to

\[
\tilde{p}_i(s_1, t) = p_i^*(s_1, d_2, t) + (a_2 - d_2)[p_i^*(s_1, d_2 + 1, t) - p_i^*(s_1, d_2, t)].
\]

We call this heuristic policy the interpolation heuristic because it linearly interpolates the Nash equilibrium prices.
This interpolation heuristic can be easily generalized to an n-firm model by replacing Eq. (9) with a summation of $2^{n-1}$ terms:

$$
\tilde{p}(s_1, t) = \sum_{s_2=d_2}^{d_2+1} \cdots \sum_{s_n=d_n}^{d_n+1} \left[ \prod_{j=2}^{n} (1(\xi_j = d_j)(d_j + 1 - a_j) + 1(\xi_j = d_j + 1)(a_j - d_j)) \right] p^*_1(s_1, s_2, \ldots, s_n, t),
$$

where $c_i$ is the initial inventory level of firm $i$, $a_i = c_i t / T$, and $d_i = \lfloor a_i \rfloor$, $i = 2, \ldots, n$, and that the indicator function $1(A)$ is equal to 1 if statement $A$ is true, and is equal to 0 otherwise.

4.2. Numerical experiments with a two-firm model

In this section, we let firm 1 use the interpolation heuristic and evaluate its performance in a two-firm model. We consider three scenarios: In the first scenario, we let firm 2 have complete information as to what firm 1 is doing, and let firm 2 maximize its own expected revenue. This scenario allows us to evaluate the interpolation heuristic when the two firms have asymmetric information. In the second scenario, we assume that neither firm has full information, and let both firms use the same interpolation heuristic. In the third scenario, we let firm 2 have complete information, and let firm 2 use the policy that minimizes firm 1’s expected revenue — called predatory pricing. This scenario provides a lower bound on the expected revenue from the interpolation heuristic.

In the first scenario, let $U_2(s_1, s_2, t)$ denote the maximized expected total revenue for firm 2 at the beginning of period $t$ if firm $i$ still has $s_i$ items in inventory, $i = 1, 2$. The recursive equation for $U_2(\cdot, \cdot, \cdot)$ is

$$
U_2(s_1, s_2, t) = \lambda \left[ \max_{p_1} (q_2(\tilde{p}(s_1, t), p_2)(p_2 + U_2(s_1, s_2 - 1, t - 1)) + q_1(\tilde{p}(s_1, t), p_2)U_2(s_1 - 1, s_2, t - 1)) + (1 - \lambda) U_2(s_1, s_2, t - 1) \right],
$$

where $\tilde{p}(s_1, t)$ is defined by Eq. (9). The boundary conditions are $U_2(\cdot, 0, \cdot) = U_2(\cdot, 0, 0) = 0$. Letting $\tilde{p}_2(s_1, s_2, t)$ denote firm 2’s optimal price solved from Eq. (10), we can calculate the expected revenue for firm 1 in state $(s_1, s_2, t)$ with the following recursive equation:

$$
U_1(s_1, s_2, t) = \lambda \left[ q_1(\tilde{p}_1(s_1, t), \tilde{p}_2(s_1, s_2, t))(\tilde{p}_1(s_1, t) + U_1(s_1 - 1, s_2, t)) + q_2(\tilde{p}_1(s_1, t), \tilde{p}_2(s_1, s_2, t))U_1(s_1, s_2 - 1, t) + q_0(\tilde{p}_1(s_1, t), \tilde{p}_2(s_1, s_2, t))U_1(s_1, s_2, t) \right] + (1 - \lambda) U_1(s_1, s_2, t - 1),
$$

The boundary conditions are $U_1(0, \cdot, \cdot) = U_1(\cdot, \cdot, 0) = 0$.

In the second scenario, neither firm can keep track of the other’s real-time inventory level, so that both firms use the same interpolation heuristic. Because either firm can compute the price by Eq. (9) beforehand, we can use a recursive equation similar to Eq. (11) to compute the expected revenue of either firm.

In the third scenario, we let firm 2 use the predatory pricing to minimize firm 1’s expected revenue. Let $W_1(s_1, s_2, t)$ denote the minimized expected total revenue for firm 1 at the beginning of period $t$ if firm $i$ still has $s_i$ items in inventory, $i = 1, 2$. The recursive equation for $W_1(\cdot, \cdot, \cdot)$ is

$$
W_1(s_1, s_2, t) = \lambda \left[ \min_{p_2} (q_1(\tilde{p}(s_1, t), p_2)(\tilde{p}(s_1, t) + W_1(s_1 - 1, s_2, t - 1)) + q_2(\tilde{p}(s_1, t), p_2)W_1(s_1, s_2 - 1, t - 1) + q_0(\tilde{p}(s_1, t), p_2)W_1(s_1, s_2, t - 1)) \right] + (1 - \lambda) W_1(s_1, s_2, t - 1),
$$

<table>
<thead>
<tr>
<th>$c_2$</th>
<th>Equilibrium $^a$ ($)</th>
<th>Heuristic vs. optimal $^b$</th>
<th>Heuristic vs. heuristic $^c$</th>
<th>Lower bound $^d$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Firm 1</td>
<td>Firm 2</td>
<td>Firm 1</td>
<td>Firm 2</td>
<td>Firm 1</td>
</tr>
<tr>
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<td>3060</td>
<td>0.9997</td>
<td>1.0002</td>
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</tr>
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<td>40</td>
<td>7160</td>
<td>6516</td>
<td>0.9949</td>
<td>1.0031</td>
</tr>
<tr>
<td>50</td>
<td>7002</td>
<td>6811</td>
<td>0.9898</td>
<td>1.0048</td>
</tr>
</tbody>
</table>

$^a$ Both firms use the prices in equilibrium when real-time inventory levels are public information.

$^b$ Firm 1 uses the interpolation heuristic, and firm 2 – knowing firm 1’s real-time inventory level – maximizes its own expected revenue.

$^c$ Both firms use the interpolation heuristic.

$^d$ Firm 1 uses the interpolation heuristic, and firm 2 – knowing firm 1’s real-time inventory level – minimizes firm 1’s expected revenue.
where \( p_1(s, t) \) is defined by Eq. (9). The boundary conditions are 
\[
W_1(0, C_1, C_1) = W_1(C_1, 0, C_1) = 0. 
\]
Firm 1’s expected revenue in this scenario \( W_1(C_1, 0, C_1) \) serves as a lower bound for the interpolation heuristic. The expected total revenues for firm 2 can be calculated by a recursive equation similar to Eq. (11).

Table 2 presents numerical results for \( c_1 = 30 \) and \( c_2 \) ranging from 10 to 50. We let \( a_1 = 1, a_2 = 0.5, b = 0.01, T = 2000, \) and \( \lambda = 0.1 \). The first two columns present the expected revenue in Nash equilibrium defined in Section 3.1. The rest of the table concerns the performance of the interpolation heuristic in scenarios 1, 2, and 3, respectively, with the results presented as the fraction of the expected revenue in Nash equilibrium. Finally, Tables 3 and 4 repeat Table 2 with \( a_2 \) changed to 1 and 2, respectively.

As seen in Tables 2–4, the interpolation heuristic works very well in the first scenario, when the two firms have asymmetric information. Even without knowing firm 2’s real-time inventory level, firm 1 still attains at least 97% of the revenue in the complete information model. Although firm 2 can benefit from firm 1’s lack of complete information, the benefit is almost negligible. These results are not surprising because the interpolation heuristic serves as a very good approximation to the price in Nash equilibrium, so firm 2’s best response is also close to its own Nash equilibrium price. Consequently, the

### Table 3

<table>
<thead>
<tr>
<th>( c_2 )</th>
<th>Equilibrium(^a) ($)</th>
<th>Heuristic vs. optimal(^b)</th>
<th>Heuristic vs. heuristic(^c)</th>
<th>Lower bound(^d)</th>
</tr>
</thead>
<tbody>
<tr>
<td>10</td>
<td>7728</td>
<td>3552</td>
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<td>1.0044</td>
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<tr>
<td>20</td>
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<td>5767</td>
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<td>7338</td>
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<td>1.0038</td>
</tr>
<tr>
<td>50</td>
<td>6924</td>
<td>9135</td>
<td>0.9817</td>
<td>1.0062</td>
</tr>
</tbody>
</table>

\(^a\) Both firms use the prices in equilibrium when real-time inventory levels are public information.

\(^b\) Firm 1 uses the interpolation heuristic, and firm 2 – knowing firm 1’s real-time inventory level – maximizes its own expected revenue.

\(^c\) Both firms use the interpolation heuristic.

\(^d\) Firm 1 uses the interpolation heuristic, and firm 2 – knowing firm 1’s real-time inventory level – minimizes firm 1’s expected revenue.

### Table 4

<table>
<thead>
<tr>
<th>( c_2 )</th>
<th>Equilibrium(^a) ($)</th>
<th>Heuristic vs. optimal(^b)</th>
<th>Heuristic vs. heuristic(^c)</th>
<th>Lower bound(^d)</th>
</tr>
</thead>
<tbody>
<tr>
<td>10</td>
<td>7727</td>
<td>4543</td>
<td>0.9982</td>
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<tr>
<td>30</td>
<td>7329</td>
<td>10297</td>
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</tr>
<tr>
<td>40</td>
<td>7112</td>
<td>12359</td>
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<td>1.0033</td>
</tr>
<tr>
<td>50</td>
<td>6881</td>
<td>14002</td>
<td>0.9780</td>
<td>1.0053</td>
</tr>
</tbody>
</table>

\(^a\) Both firms use the prices in equilibrium when real-time inventory levels are public information.

\(^b\) Firm 1 uses the interpolation heuristic, and firm 2 – knowing firm 1’s real-time inventory level – maximizes its own expected revenue.

\(^c\) Both firms use the interpolation heuristic.

\(^d\) Firm 1 uses the interpolation heuristic, and firm 2 – knowing firm 1’s real-time inventory level – minimizes firm 1’s expected revenue.

### Table 5

<table>
<thead>
<tr>
<th>( c_2 )</th>
<th>Equilibrium(^a) ($)</th>
<th>Heuristic vs. optimal(^b)</th>
<th>Heuristic vs. heuristic(^c)</th>
<th>Lower bound(^d)</th>
</tr>
</thead>
<tbody>
<tr>
<td>10</td>
<td>4926</td>
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</tr>
<tr>
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<td>4358</td>
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</tr>
<tr>
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<td>4343</td>
<td>2960</td>
<td>0.9861</td>
<td>1.0429</td>
</tr>
<tr>
<td>50</td>
<td>4342</td>
<td>2960</td>
<td>0.9861</td>
<td>1.0429</td>
</tr>
</tbody>
</table>

\(^a\) Both firms use the prices in equilibrium when real-time inventory levels are public information.

\(^b\) Firm 1 uses the interpolation heuristic, and firm 2 – knowing firm 1’s real-time inventory level – maximizes its own expected revenue.

\(^c\) Both firms use the interpolation heuristic.

\(^d\) Firm 1 uses the interpolation heuristic, and firm 2 – knowing firm 1’s real-time inventory level – minimizes firm 1’s expected revenue.
expected revenues for both firms are close to those in Nash equilibrium. These numerical experiments demonstrate the effectiveness of the interpolation heuristic.

In the second scenario, neither firm has complete information and both firms use the interpolation heuristic. Once again, because the interpolation heuristic approximates the equilibrium price in the complete information model well, the expected revenues of both firms are very close to those in the complete information model.

In the third scenario, the lower bound on the interpolation heuristic is very encouraging as seen in the last column of Tables 2–4. In these examples, this lower bound guarantees firm 1’s expected revenue to be consistently well above 95% of the revenue in Nash equilibrium even when firm 2 uses predatory pricing. In order to minimize firm 1’s expected revenue, most of the time firm 2 uses a very low price to prevent the customer from buying firm 1’s products. Consequently, as \( c_2 \) increases, firm 2 can take more customers away, which makes it more difficult for firm 1 to sell an item – even if firm 1 uses the Nash equilibrium price. In other words, the difference between the equilibrium revenue and this lower bound is mostly due to the extremely low prices posted by firm 2, rather than due to the difference between our heuristic policy and the equilibrium price.

Whereas Tables 2–4 report results when the interpolation heuristic is supposed to work well, Tables 5–7 report results for \( T = 1000 \) so that \( \lambda T \) is in the same order as \( c_1 + c_2 \). As seen in these tables, the expected revenues in the first scenario (firm 2 has complete information and maximizes its own expected revenue) and in the second scenario (firm 2 has incomplete information and uses the same heuristic) are still close to those in the complete information model, because when the chance of selling out is small, both firms would use a price close to its myopic price defined in Section 3.1. If the goal of firm 2 is to hurt firm 1, however, then firm 2 can do much damage by lowering its price substantially, as seen in the last column of Tables 5–7. In practice, unless firm 2 uses predatory pricing, we would expect the interpolation heuristic to produce satisfactory results in general.

### 5. Conclusions

In this paper, we propose a game-theoretic model to describe the dynamic price competition between firms. Upon arrival, a customer compares the current product prices and will purchase either his most preferable product or not at all.
Using the multinomial logit model to describe a customer’s discrete choice, we show the existence of a Nash equilibrium. We illustrate numerically how a firm can benefit by taking into account the competition when setting its product price, and also develop a heuristic policy that does not require real-time inventory levels of the other firms. The findings in this paper provide managerial insights into dynamic price competition in revenue management.

One assumption we made implicitly by choosing the MNL choice model is the axiom of independence of irrelevant alternatives. In other words, when a firm’s inventory is depleted, according to the MNL model all the other firms will benefit equally with their probability of sale increasing with the same proportion. Whereas this assumption may be reasonable in industries if the coefficient $z$ is tied with a firm’s popularity (such as airlines), it may not be applicable to other industries if $z$ is tied with the quality of a product. In the latter case, the depletion of one firm’s inventory would bring more benefit to those firms that sell products of comparable quality. A different customer choice model is necessary if the axiom of independence of irrelevant alternatives does not hold.

In addition to a different discrete choice model, there are a few possible future research directions. First, allowing batch demand is an important extension because in many cases a customer needs to buy multiple units of the same product. Second, if a firm can pay a cost to find out the number of items each competitor still has at any time, then whose inventory level is most valuable to learn about, how much is a firm willing to pay for this information, and when should a firm do it? The answers to these questions will further help a manager improve revenue management in a competitive market.

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**References**


