Decision Support

A competitive dynamic pricing model when demand is interdependent over time

Soheil Sibdari a,*, David F. Pyke b

a Charlton College of Business, University of Massachusetts, North Dartmouth, MA 02747, USA
b School of Business Administration, University of San Diego, San Diego, CA 92110, USA

ARTICLE INFO

Article history:
Received 12 December 2008
Accepted 3 March 2010
Available online 21 March 2010

Keywords:
Pricing
Revenue management
Game theory
Discrete choice model

ABSTRACT

In this study, we contribute to the dynamic pricing literature by developing a finite horizon model for two firms offering substitutable and nonperishable products with different quality levels. Customers can purchase and store the products, even if they do not need them at the time, in order to use them in future. The stockpile of the products generated by customers affects the demand in future periods. Therefore, the demand for each product not only is a function of prices and quality levels, but also of the products’ stockpile levels. In addition, the stockpile levels change the customers’ consumption behavior; more product in a stockpile leads to more consumption. Therefore, we address not only the price and demand relationship but also the stockpiling and consumption relationship in a competitive environment.

The decision variable of each firm at the beginning of each period is its unit sale price. We use a deterministic dynamic program to calculate the equilibrium prices at the beginning of each period. By assuming that the market stockpile is public information, we show the existence of a unique Nash equilibrium. We next consider the case when the firms do not know the market stockpile. We then develop appropriate heuristics to calculate the optimal prices in each case. A numerical study is also provided to calculate the price levels in different scenarios and compare their performances.

1. Introduction and overview

Revenue management is a technique that has been widely used in travel industries, such as airlines, rental cars, hotels, cruise lines, and railroad services (Bitran and Mondshein, 1995; Gallego and van Ryzin, 1997). More recently, using the new advances in information technology, other sectors have employed revenue management to improve revenues and service and to reduce costs. Firms such as GM, John Deere, and Radio Shack have pursued these techniques fairly aggressively (Hall and Kopalle, 2009). However, due to the nature of its applications in travel industries, some restrictive assumptions have been made in the dynamic pricing literature. For example, it is often assumed that a potential customer requests only one unit of the product during the sale horizon (a passenger needs only one seat for a particular flight). In addition, it is assumed that potential customers do not act strategically and that they do not adjust the timing of their purchase. Using these assumptions, a customer makes her purchase decision solely based on current prices and ignores the opportunity of purchasing the product at a possible discounted price in future. Therefore, in the majority of studies in dynamic pricing, it is assumed that demand is independent across periods and that price levels in the current period do not affect demand in the future. If products are nonperishable this assumption should be relaxed. In a period with a price promotion, a customer may purchase more units than her needs and store the remainder for future use. In this case, a price promotion leads to high demand in the current period and low demand in the following periods. Therefore, in industries with nonperishable products, the demand for each product is not only a function of current prices but also a function of prices charged in previous periods.

Another important pricing issue for nonperishable products is the consumption effect. In the marketing literature, the impact of promotions on product consumption has been well addressed (see Ailawadi and Neslin, 1998; Assuncao and Meyer, 1993). Customers may adjust their consumption based on their own stockpile levels: more products on the shelf lead to higher consumption. Thus, the pricing decision not only creates a dependency of demand across periods, but also affects consumers’ consumption. Fleischmann et al. (2006) modeled this situation in a deterministic, monopolistic environment.

Another restrictive assumption in the majority of research in dynamic pricing is the assumption of a monopolistic firm. In such monopolistic models, demand depends solely on the price posted by the monopolist. In part because of the wide availability of information over the internet, this assumption no longer seems valid because customers can observe different products, compare their quality and prices, and choose the one that best matches their needs.

* Corresponding author. Tel.: +1 508 999 8019; fax: +1 508 999 8646.
E-mail addresses: ssibdari@umassd.edu (S. Sibdari), davidpyke@sandiego.edu (D.F. Pyke).

© 2010 Elsevier B.V. All rights reserved.
In addition, a promotional price offered by one firm increases the market stockpile of that product, which in turn affects the demand of all competitive products. Consequently, the demand of a product depends not only on its quality, price and stockpile level, but also on the quality, price and stockpile level of competitive products.

In this paper, we relax three commonly employed assumptions of the dynamic pricing literature. First, we allow for purchases of more than a single unit, with the ability of customers to store product over time. Second, we model the consumption effect, and third, we consider a competitive environment. Although some papers in the current literature have addressed one or two of these issues, we incorporate all three.

1.1. Related research

Research on dynamic pricing has grown rapidly since the deregulation of the US airline industry in the 1970s. The earliest work on dynamic pricing appears to be that of Kincaid and Darling (1963). Monopolistic dynamic pricing has been well studied in literature, including papers by Gallego and van Ryzin (1994), Gallego and van Ryzin (1997), Chatwin (2000), Lin (2004), Feng and Xiao (2000), and Zhao and Zheng (2000). In this stream of research the goal is to manage the sale of a given number of items in a given period of time in order to maximize expected revenue. The main decision variable is the unit price. In most cases, the optimal policy has been addressed, analytically or numerically, and structural properties of the optimal policy have been studied.

A similar problem setting, but with more than one firm, leads to a new stream of research known as competitive dynamic pricing. In this case, an oligopolistic model has been used to address the competition between multiple firms in the market when, again, price is the decision variable. Since there is more than one firm in the market, the optimal policy of each firm depends on other firms’ actions, often giving rise to a Nash equilibrium: Dudgeon (1992), Li and Oum (1997), Perakis and Sood (2004), and LIN and Sibdari (2009). In all of the above papers, the demand is independent over time, i.e. demand in the current period is independent of demand in previous periods.

Evidence for the dependency of demand across periods is abundant. For instance, as noted by Neslin (2002), a promotion in the current period not only generates a spike in demand in that period but also reduces demand in subsequent periods (see also Pauwels et al., 2002; Srinivasan et al., 2004). Customers may forward-buy, which causes an increase in demand during the promotion. Neslin (2002) explains this phenomenon in terms of reference prices and stockpiling and argues that the magnitude of the effect depends on the specific product category. Reference prices refer to the fact that demand often depends not only on current price, but also on a reference price that reflects past prices (see Lattin and Bucklin, 1989). We do not explicitly model reference prices in this paper.

Many products exhibit the stockpiling effect, including most nonperishable products, such as paper towels, potato chips, ice cream, and beauty products (Blattberg et al., 1981; Neslin, 2002; Fleischmann et al., 2006; Slonim and Garbarino, 2008; Popescu and Wu, 2007). Even some perishable products, such as yogurt, are stockpiled if the time for consumption is shorter than the life of the product. Stockpiling and forward buying have been demonstrated empirically by many authors, including Mace and Neslin (2004), Bell et al. (1999) and Bell et al. (2002). Slonim and Garbarino (2006) combine both reference price and stockpiling effects in their research. They employ simulation of empirical data and analytical models to show that stockpiling may explain the apparent greater sensitivity of demand to price decreases than to price increases.

It is clearly difficult to have complete knowledge of individual consumers’ stockpiles or reference prices. However, some researchers have made progress in estimating consumer stockpiles as a function of past prices and demand (Ailawadi and Neslin, 1998). Other researchers have helped us understand reference prices in more detail (Greenleaf, 1995; Kopalle et al., 1996; Natter et al., 2007; Popescu and Wu, 2007).

In addition to competition and the dependence of demand across periods, we model the consumption effect. Ailawadi and Neslin (1998) show that higher levels of inventory on the shelf can lead to higher levels of consumption, although the magnitude of this effect depends on product category. See also Assuncao and Meyer (1993), Bell et al. (1999), and Gourville and Soman (2002).

1.2. Overview and outline

In this paper, we contribute to the dynamic pricing literature by developing a model that is applicable beyond the airline and hospitality industries. We relax the restrictive assumption of demand in a given period being independent from the prices charged in other periods. Relaxing this assumption allows one to apply dynamic pricing methods for markets of nonperishable products. We develop a game-theoretic model over a finite time horizon to describe the competition between two firms operating in such markets. Finally, we incorporate the consumption effect that has often been ignored in the literature.

In our model, the total number of potential customers (or market size) is known by both firms. Since the products are nonperishable, customers can store products and therefore can manage their consumption. In each period, we define the market stockpile as the total number of products that have been stored by all customers in the market and not consumed in that period. In keeping with the marketing literature, lower stockpiles at the end of a given period lead to higher demand in the next period. We use aggregate demand in the sense that when market stockpile is low the total number of potential customers is high. In each period, customers demand only one unit of product, but they accumulate their stockpile by being active in multiple periods. Therefore, the demand in each period depends on the market stockpile and on the current and previous prices set by the firms.

We assume that there is no limit on production capacity and that the firms can satisfy any amount of demand (naturally up to the market size). The goal of each firm is to maximize its own expected total profit over the finite horizon. We examine two cases in detail. First, we assume that both firms have real-time information about the exact market stockpile (full information assumption). In this case, we show the existence and uniqueness of a Nash equilibrium and we use a dynamic programming technique to calculate the equilibrium price levels. We also study the relationship between the equilibrium prices and other parameters of the model such as the quality and stockpile levels. Second, we relax the full information assumption and use a heuristic to calculate the price levels. The heuristic policy estimates real-time market stockpile using the prices that have been charged in previous periods. We use numerical examples to evaluate this heuristic.

The rest of this paper is organized as follows: in Section 2, we develop our game-theoretic model. To simplify our theoretical analysis we provide a single-period model in Section 3. Section 4 examines the dynamics for a finite horizon. Section 5 provides some numerical insights about the importance of considering price competition. In Section 6 we provide a heuristic for the case when firms do not know the real-time market stockpile. Section 8 provides concluding remarks.

2. Model

Consider two firms, firm 1 and firm 2, that sell their products over a finite time horizon. The products of these firms have
different quality levels, described by the parameter \( \alpha_i, i = 1, 2 \). We assume that the firms face an admittedly simple production cost \( K_i(x) = k_i x \), where \( x > 0 \) is the number of products and \( k_i > 0 \) is the marginal cost of production for firm \( i, i = 1, 2 \). The firms do not face capacity constraints and thus they can fulfill any amount of demand. We employ these assumptions for tractability so that we can generate insights into this complex problem.

The demand is interdependent over time in that the prices in the current period affect future demand levels. We model this by assuming that the total population is fixed, and since the products can be stored to be used in the future, higher sales volume today leads to lower demand in the future and vice versa. Therefore, the demand interdependency is modeled by tracking the aggregate stockpile of the products stored by customers. Because the products are substitutes for each other, the demand for each product not only depends on its own stockpile but also depends on the other product’s stockpile. Let \( m_t \) is the stockpile of product \( i \) at the end of period \( t \). There are a total of \( N \) individuals in the market, where \( N \) is constant and independent from the other parameters of the model. Among this population, in period \( t \), \( N_t(m_t^{-1}) \) individuals are active and plan to purchase one unit of the product. A higher market stockpile leads to a lower number of potential customers; i.e., \( \frac{dN_t}{dm_t} < 0 \). Note that we aggregate the total demand and that we do not consider the case where a specific customer decides to purchase multiple items. Customers accumulate their stockpile by purchasing one unit of the product in multiple periods.

The potential customers observe the products’ price and quality levels and decide to buy one of the products or leave empty handed. For each customer we define \( q_i(p_1, p_2), i = 0, 1, 2 \) as the probability of purchasing from firm \( i \), where \( i = 0 \) means that customers leave empty handed and \( p_i \) is the price charged by firm \( i \). However, \( q_i(p_1, p_2) \) can also be viewed as the proportion of customers purchasing product \( i \). In this paper, since we use deterministic dynamic programming, we refer to \( q_i(p_1, p_2) \) as proportion of customers purchasing product \( i \). Since customers face a discrete choice of sellers, we use the multinomial logit (MNL) model to describe this choice. The MNL model is commonly used in the marketing literature (see Lilien et al., 1992; Anderson and Palma, 1992; Franses and Paap, 2001). In this model, it is assumed that customers act independently from each other to maximize their own utility. If a customer chooses product \( i \) at price \( p_i \), then his or her utility is equal to

\[
U_j = \alpha_i - \beta p_i + Z_i, \quad i = 1, 2
\]

and the utility of a customer who leaves empty handed is

\[
U_0 = Z_0.
\]

In this utility function, \( \alpha_i \) describes the quality of firm \( i \)’s product, and \( \beta \) captures the response to the price level. In the MNL model, the idiosyncratic preference of each customer is described by the random variables \( Z_i, i = 0, 1, 2 \). These random variables are independent and identically distributed (i.i.d) Gumbel (double exponential) random variables with mean 0 and shape parameter \( \mu \). The cumulative distribution function for the Gumbel distribution is (see Ben-Akiva and Lerman (1985) for derivation):

\[
P(Z_i \leq z) = e^{-e^{-z}}, \quad z \geq 0,
\]

where \( \gamma \approx 0.5572 \) is Euler’s constant. Using the Gumbel distribution, the purchase probabilities from firm \( i \) can be calculated from the following closed form equation:

\[
q_i(p) = P(U_i = \max_{j=0,1,2} U_j) = \frac{e^{\gamma_i - \beta p_i}}{1 + \sum_{j=1}^{3} e^{\gamma_j - \beta p_j}}, \quad i, j = 1, 2 \text{ and } i \neq j
\]

and the proportion of customers who do not purchase is

\[
q_0(p) = \frac{1}{1 + \sum_{j=1}^{3} e^{\gamma_j - \beta p_j}}, \quad (2)
\]

where \( p = (p_1, p_2) \) is the price vector set by the firms. In this equation, the positive parameter \( \mu \) is used to model horizontal differentiation between products. For higher values of \( \mu \), the horizontal differentiation plays a more important role, and as \( \mu \) increases the consumers become more heterogeneous and less responsive to the products’ price. For \( \mu = 0 \) all consumers choose the option with the higher utility in which case the horizontal differentiation is eliminated. We set \( \mu = 1 \), since this parameter can be absorbed into the constants \( \alpha_i, i = 1, 2 \), and \( \beta \). \(^1\)

We assume that the available price set is \( p_i \in [0, P] \), where \( P \) is a high price such that \( q_i(P) = q_i(0) \approx 0 \). Note that \( q_i(\infty) = q_i(0, \infty) = 0 \). To determine the total demand per period, we use a multiplicative model (see Vives, 1999). In period \( t \), total demand for product \( i \), for price vector \( p = (p_1, p_2) \), is:

\[
D'_i(p, m^{-1}) = N_t(m^{-1}q_i(p_1, p_2)) \cdot i, 0, 1, 2,
\]

where the total number of customers leaving empty handed is \( N_t(m^{-1}q_0(p_1, p_2)) \). Note that the first derivatives of \( D'_i(p, m^{-1}) \) with respect to \( p_1 \) and \( p_2 \) are \(-\beta q_i(1 - q_i) < 0 \) and \(-\beta q_i q_j < 0 \) which confirms that the demand of firm \( i \) is non-increasing in \( p_i \) and non-decreasing in \( p_j \) for \( i, j = 1, 2 \) and \( i \neq j \).

The market stockpile is consumed according to a consumption function \( C'(m_t, m^{-1}) = C'(m^{-1} + D'_2(p, m^{-1})) \) which allows the demand in the current period to be consumed in the same period. We assume that the products are substitutable and have “roughly” equal quality, which means that customers can choose either one if they desire. Using this assumption, we can track only one total market stockpile instead of tracking both market stockpiles.

Considering \( C'_i(x) \) to be the consumption function of product \( i \), the consumption function at the end of period \( t \) is \( C'_i(x) = C'_i(x) + C'_j(x) \). Note that \( C'_i(x) \) is non-decreasing with respect to \( x \) for all \( x > 0 \).

The state transformation function is:

\[
m_t(p, m^{-1}) = m_t^{-1} + D'_1(p, m^{-1}) + D'_2(p, m^{-1}) - C'(m^{-1}). \quad (4)
\]

In each period, the order of the events can be described as follows.

1. At the beginning of period \( t \), the firms observe the market stockpile.
2. The firms simultaneously determine their production levels and product prices.
3. The demand for each product occurs as a function of the prices, the quality levels, and the market stockpile.
4. The post-demand consumption occurs based on the new market stockpile, thus creating the market stockpile for the next period.

3. Single-period game

In this section, we study our model in the last period of the finite horizon, examining it in a single-period game. Both firms desire to maximize their immediate expected profit and do not take into account how the potential sale might affect the market stockpile or their future profit. Therefore, we designate the firm’s equilibrium price in this single-period model as their myopic price.\(^1\)

Absorbing \( \alpha_i = 1 \) into \( z, i = 1, 2 \), and \( \beta \) transfers the idiosyncratic preference of customers with respect to product \( i \) into idiosyncratic preference of customers with respect to quality and price. However, one can think of a fixed \( \mu = 1 \) as a constraint that the model considers the aversive and conservative customers. However, even with this point of view since the focus of our paper is to study impact of price, it is not crucial to the results. We thank one of the reviewers for highlighting this issue.
Let \( \mathbf{p} = (p_1, p_2) \) denote the joint price vector at the beginning of period \( T \) (the last period). Recall \( D_i^t(p, m^{t-1}) = q_i(p)N^t(\mathbf{m}^{t-1}) \) and \( K(D_i) = kD_i \). The profit function for the last period for firm \( i \) is

\[
\pi_i(p^t, m^{t-1}) = \pi_i - D_i(p, m^{t-1}) - K(D_i(p, m^{t-1})) = (p_i - k) \cdot q_i(p)N^t(\mathbf{m}^{t-1}) \quad \text{for } i, j = 1, 2 \text{ and } i \neq j.
\]

(5)

Since there is no fixed cost of production, both firms produce what they can sell; i.e. they do not benefit from cost saving. A Nash equilibrium arises if each firm’s price is the best response to the prices posted by the other firm, so that no firm has an incentive to change its price. In other words, a joint policy \( \mathbf{p} = (p_1, p_2) \) would be a Nash equilibrium if:

\[
\pi_i(p^t, m^{t-1}) = \max_{p_i} \pi_i(p, p_j, m^{t-1}) \quad i, j = 1, 2 \text{ and } i \neq j.
\]

(6)

We now show that the profit function \( \pi_i(p, m^{t-1}) \) is quasi-concave in \( p_i \).

**Lemma 1.** The immediate profit \( \pi_i(p, m^{t-1}) \) in Eq. (5) is strongly quasi-concave in \( p_i \).

**Proof.** Differentiating \( \pi_i(p) \) with respect to \( p_i \) yields

\[
\frac{\partial \pi_i(p)}{\partial p_i} = N(t)(1 - \beta p_i + \frac{1}{2} p_i - k)(1 - \gamma p_i).
\]

(7)

Note that the function \( h(p) = 1 - \beta p_i - (1 - q_i(p)) \) is strictly decreasing in \( p_i \), and that \( h(0) = 1 - \beta k(1 - q_i(p)) > 0 \) and \( h(T) = 1 - \beta T - k < 0. \) Since \( T \geq k \), there exists a unique solution \( 0 < p_i < T \) such that \( h(p_i) = 0 \). It then follows that \( \pi_i(p, m^{t-1}) \) is quasi-concave in \( p_i \), because \( \pi_i(p, m^{t-1}) \) increases when \( p_i < p_i \) reaches its maximum when \( p_i = p_i \), and then decreases when \( p_i > p_i \). □

From firm \( i \)'s standpoint, we define \( p_i \) as the price set by the other firm. Based on Lemma 1, for a given \( p_i \), we define

\[
\phi_i(p_i) = \arg \max_{p_i} \pi_i(p_i, m^{t-1})
\]

(8)

as firm \( i \)'s best response function. Given the definition of the allowed price set, \( \phi_i(p_i) \) is uniformly bounded in \([0, T]\) for all \( p_i \).

**Theorem 1.** There exists a pure-strategy Nash equilibrium \( \mathbf{p}^* \) characterized by Eq. (6).

**Proof.** The existence of a Nash equilibrium follows from Theorem 2.1 in Vives (1999)—originally attributed to Debreu (1952)—which states that a Nash equilibrium exists if the strategy sets are non-empty convex and compact, and the payoff to firm \( i \) is continuous in the actions of all firms and quasi-concave in its own action. In our single-period model, this claim is true because according to the definition of the allowed price set, firm \( i \) does not need to consider any price greater than \( T \) when finding its best response. In addition, the profit function \( \pi_i(p) \) is continuous in \( p_i \); \( i, j = 1, 2 \) and \( i \neq j \), and by Lemma 1 quasi-concave in \( p_i \). Consequently, a Nash equilibrium exists.

To prove the uniqueness of the Nash equilibrium \( \mathbf{p}^* \), first observe that because the logarithm is a strictly increasing function, the best response functions defined in Eq. (8) do not change if the profit of each firm becomes \( \log \pi_i(p_i) \), \( i = 1, \ldots, n \). In other words, the game with profit functions \( \pi_i(p_i) \), \( i = 1, 2 \), has a unique Nash equilibrium if and only if the game with profit functions \( \log \pi_i(p_i) \), \( i = 1, 2 \), also has a unique Nash equilibrium. The latter can be verified by the sufficient condition given in Theorem 5 in Cachon and Netessine (2004) and its restatement on page 152 of that reference (often known as the “diagonal dominance” condition):

\[
\sum_{j=1}^n \frac{\partial^2 \log \pi_i(p_i)}{\partial p_j} < \frac{\partial^2 \log \pi_i(p_i)}{\partial p_i^2}.
\]

(9)

Substituting the derivatives from Eq. (6) into the preceding yields

\[
(\beta^2 q_i(p) q_i(p)) - \left( \frac{1}{|p_1 - k|} + \beta^2 q_i(p)(1 - q_i(p)) \right)
= - \frac{1}{(p_1 - k)^2} - \beta^2 q_i(p) q_i(p) < 0.
\]

Therefore, the game with payoff functions \( \log \pi_i(p) \), \( n = 1, 2 \), has a unique Nash equilibrium, and so has the game with payoff functions \( \pi_i(p) \), \( i = 1, 2 \). □

**4. Dynamics in period 1 through \( T - 1 \)**

We have shown that a single-period game has a unique Nash equilibrium. In this section, we study the problem over a finite period of time that starts at period 1 and ends at the end of period \( T \). At the beginning of period \( t \), the net profit of firm \( i \) is given by:

\[
\pi_i^t(p^t, m^{t-1}) = p_i D_i(p^t, m^{t-1}) - K(D_i(p^t, m^{t-1})) + \gamma V_t^{t+1}(m^{t}).
\]

(10)

where \( \gamma \) is the discount factor. We define \( V_t^{t+1}(m^t) \) as the continuation equilibrium profit for firm \( i \) at the beginning of period \( t + 1 \) if the total market stockpile is \( m^t \) and both firms use an equilibrium price strategy from period \( t + 1 \) onward.

Recall the \( \mathbf{p}^* = (p_1, p_2) \) would be a Nash equilibrium if for \( i, j = 1, 2 \), and \( i \neq j \):

\[
\pi_i^t(p^t, m^{t-1}) = \max_{p_i} \pi_i(p, p_j, m^{t-1}),
\]

(11)

where \( p_i \) is the Nash equilibrium price charged by firms \( i \)'s opponent. The best response function of firm \( i \) is:

\[
\phi_i(p_i) = \arg \max_{p_i} \pi_i(p_i, p_j, m^{t-1}) + \gamma V_t^{t+1}(m^t) \quad \text{for all } p_i \geq 0.
\]

(12)

We show the existence and uniqueness of a Nash equilibrium assuming that \( p_i \) is a continuous and bounded function with respect to \( m^t \). This assumption is reasonable since if \( p_i(m^t) \) is not a bounded function there should be an \( m^t \) for which \( p_i(m^t) = \infty \), in which case \( D_i(m^t) = q_i(p)N^t(m^t) = 0 \), since \( q_i(\infty, p) = 0 \). In other words, there is a stockpile level for which the firms set their prices to infinity to prevent customers from buying their products. Since the total number of potential customers is fixed, a firm that sets an infinite price and therefore drives its demand to zero, would only force customers to buy its competitor’s product.

Now, without loss of generality, we consider the game from firm \( i \)'s point of view. Suppose firm \( i \) uses the following pricing strategy:

\[
p_i = f(m^t),
\]

(13)

where \( f(\cdot) \geq 0 \) is a continuous and bounded function. Then, the best response function of firm \( i \) becomes:

\[
\phi_i(f(m^t)) = \max_{p_i} \pi_i(p_i, f(m^t), m^{t-1}) + \gamma V_t^{t+1}(m^t).
\]

(14)

From Eqs. (1), (3), and (4) we can conclude that:

\[
q_0(p_1, p_2) = 1 - g(m^t, m^{t-1}),
\]

(15)

where

\[
g(m^t, m^{t-1}) = m^t - m^{t-1} + C(m^t - m^{t-1}).
\]
Rearranging Eq. (15), and using Eq. (2), we can calculate \( p_1^* \) as a function of \( m^t \) and \( m^{t-1} \) as follows.

\[
p_1^*(m^t, m^{t-1}) = \frac{x_1}{\beta} - 1 - \frac{1}{\beta} \ln \left( 1 - \frac{g(m^t, m^{t-1})}{g(m^t, m^{t-1})} \right) \\
= \frac{x_1}{\beta} - 1 - \frac{1}{\beta} \ln \left( \frac{1 - g(m^t, m^{t-1})}{1 - g(m^t, m^{t-1})} \right) \tag{16}
\]

**Lemma 2.** \( p_1^*(m^t, m^{t-1}) \) in Eq. (16) is a monotonic function with respect to \( m^t \).

**Proof.** Using Eq. (15), since \( q_0(p_1, m^{t-1}) \) is a non-decreasing function with respect to \( p_1 \) (since \( \frac{\partial q_0}{\partial p_1} = \beta q_0 q_1 \geq 0 \)), it reaches its maximum at \( p_1 = \overline{p} \), and its minimum value at \( p_1 = 0 \). On the other hand, \( 1 - g(m^t, m^{t-1}) \) is a non-increasing with respect to \( m^t \) (since \( \frac{\partial g}{\partial m^t} = \frac{1}{N(m^{t-1})} < 0 \)), and therefore \( p_1^*(m^t, m^{t-1}) \) is a monotonic function with respect to \( m^t \). The maximum value of \( m^t \) will be reached at \( p_1 = 0 \) and the minimum value will be reached at \( p_1 = \overline{p} \).

We now show that, under the allowable prices, \( m^{t-1} \) gets mapped from \([0, \overline{m}] \) to a finite range. To determine the new range for \( m^{t-1} \) we use Eq. (15), and we define \( MT_1 \) and \( MT_2 \) as the new lower and upper bounds of \( m^{t-1} \). Using simple algebra, \( MT_1 = N'(m^{t-1}) (1 - q_0(\overline{p}, f(m^{t-1}))) + m^{t-1} - C'(m^{t-1}) \) and \( MT_2 = N'(m^{t-1})(1 - q_0(0, f(m^{t-1}))) + m^{t-1} - C'(m^{t-1}) \).

**Theorem 2.** There exists a pure-strategy Nash equilibrium for the multi-period game characterized by Eq. (10).

**Proof.** Since \( m^t \) is a monotonic function with respect to \( p^t \) (based on Lemma 2), we can rewrite the profit function of firm 1 only as a function of \( m^{t-1} \) and \( m^t \) as follows.

\[
V_1^t(m^{t-1}) = \max_{m^t} \left[ p_1^*(m^t, m^{t-1}) - k_1 N'(m^{t-1}) q_1(p_1^*, m^{t-1}) \right] f(m^{t-1}) + \gamma V_1^{t+1}(m^{t-1}), \tag{17}
\]

where \( p_1^*(m^t, m^{t-1}) \) can be substituted from Eq. (16). In addition, since \( 1 - g(m^t, m^{t-1}) = q_0(p_1, m^{t-1}) \), the denominator of the inner logarithm can be rewritten as \( 1 - q_0 + q_2 > 0 \) and therefore Eq. (16) is continuous. Since other terms are also continuous, Eq. (17) is continuous.

The maximization problem (17) thus maps a continuous and bounded function in the interval \([0, \overline{M}] \) into a bounded function in the interval \([MT_1, MT_2] \). It also satisfies monotonicity and discounting. Therefore, it satisfies the sufficient conditions for Blackwell’s Contraction Theorem (Blackwell, 1951) and thus the value function \( V_1^t(m^{t-1}) \) has a unique solution, which suffices for the multi-period game to have a unique Nash equilibrium.

We have now shown that there exists a Nash equilibrium for both the single-period and finite horizon games.

5. The effect of price competition

The results of the game-theoretic dynamic pricing model raise the question of the consequences of a firm ignoring the competition from other firms and instead using a strategy as if it enjoyed a monopoly in the market.

First, consider a monopoly case where firm 1 is the only firm in the market. Because there is only one firm in the market, the proportion of customers who purchase one unit of firm 1’s product at price \( p_1 \) is

\[
q_1(p_1) = \frac{e^{\beta_1 p_1}}{1 + e^{\beta_1 p_1}}
\]

and the proportion of customers who leave handed-off is 1 - \( q_1(p_1) \), according to our MNL choice model in Eqs. (1) and (2).

To maximize the expected total profit for firm 1, let \( U_1^t(m^{t-1}) \) denote the expected additional profit if the market stockpile is \( m^{t-1} \) items and there are \( t \) time periods remaining. Firm 1 can find its optimal dynamic pricing strategy by solving the following dynamic program:

\[
U_1^t(m^{t-1}) = \max_{p_1} \left[ p_1 - k q_1(p_1) N'(m^{t-1}) + \gamma U_1^{t+1}(m^t) \right], \tag{18}
\]

where the state transformation function is

\[
m^t = m^{t-1} + D_t(p^t, m^{t-1}) - C(m^{t-1})
\]

with boundary condition \( U_1(0) = 0 \). Let \( p_1^t(m^{t-1}) \) denote the maximizer of the preceding dynamic program; in other words, \( p_1^t(m^{t-1}) \) is firm 1’s optimal price when the stockpile of firm 1 product is \( m^{t-1} \) with \( t \) time periods remaining. In the rest of the paper, we will refer to this optimal strategy as the monopolistic strategy. From firm 1’s standpoint, the monopolistic strategy is easy to compute, as well as its profit function \( U_1(m^{t-1}) \). As an example, we let \( x_1 = 1, \beta = 0.01, \gamma = 1, \) and \( T = 50 \). Let the total number of customers in the market be \( \overline{M} = 200 \), so that in each period \( N'(m^{t-1}) = 200 - m^{t-1} \) of them are potential customers. The production function is linear in \( D_1; K_1(D_1) = 0.2D_1 \). Customers consume product 1 according to equation \( C(m^t) = 0.5m^t \). With this setting, and using Eq. (18), iteratively we can compute \( U_1^t(200) = 25264.73 \).

Now suppose firm 2 introduces a product that directly competes with firm 1’s product, but firm 1 ignores the competition and still uses the monopolistic strategy. From firm 2’s standpoint, a rational strategy is to maximize its own expected profit. Let \( U_2^t(m_1^{t-1}, m_2^{t-1}) \) denote the expected profit for firm 2 with \( t \) time periods remaining, if product stockpiles are \( m_1^{t-1} \) and \( m_2^{t-1} \). We can find the optimal policy for firm 2 by solving the following dynamic program:

\[
U_2^t(m_1^{t-1}, m_2^{t-1}) = \max_{p_1, p_2} \left[ p_2 - k q_2(p_2) N'(m_1^{t-1}) + \gamma U_2^{t+1}(m_1^t, m_2^t) \right] + \gamma U_2^{t+1}(m_1^t, m_2^t) \tag{19}
\]

with boundary conditions \( U_2(t, 0) = 0 \) and the following state transformation functions.

\[
m_1^t = m_1^{t-1} + D_t(p^t, m_1^{t-1}) - C(m_1^{t-1})
\]

for \( i = 1, 2 \).

Now, consider the same example in the presence of firm 2. Assume the production functions are the same; \( K_i(D_i) = 0.2D_i \). The customers consume both products according to the function \( C_i(m_i^t) = 0.5m_i^t \) and the total market stockpile is consumed using equation \( C(m^t) = 0.5m^t \), where \( m^t = m_1^t + m_2^t \). Other parameters take the same values. Table 1 shows the expected profit for both firms if firm 1 uses the monopolistic strategy and firm 2 uses the corresponding optimal policy defined by Eq. (19). Note that the absence of firm 2 is equivalent to the case when \( x_2 = -\infty \). In this case \( q_1(p_1) = \frac{e^{\beta_1 p_1}}{1 + e^{\beta_1 p_1}} \), \( q_2 = 0 \), and firm 1 enjoys a monopoly.

Columns 2 and 3 present the expected profit of both firms in the Nash equilibrium defined in Section 4. Columns 4 and 5 present the performance of the monopolistic policy compared to the Nash equilibrium. To calculate the Nash equilibrium, we use the tâtonnement process, which has been well studied in the economics literature. The tâtonnement process can be viewed as a hierarchical process consisting of two optimization steps. In each step, using
a pre-determined best response function, the optimal strategy of a player is calculated using the most recent action taken by its opponent. This process continues until the benefit of no player is improved, in which case the Nash equilibrium is reached.

As seen in the last two columns of Table 1, if the product quality of firm 2 is much lower than that of firm 1, i.e., $q_2 < q_1$, then firm 1 still dominates the market and therefore the monopolistic strategy still performs well for firm 1 (it can generate up to 96% of the equilibrium profit). On the other hand, as the quality of firm 2 gets better, then by using the monopolistic strategy, firm 1 tends to set the price too high considering its relative quality, which will lead the customer to purchase from firm 2. Consequently, firm 1’s expected profit with the monopolistic strategy becomes much lower than it would have expected.

The fact that with the monopolistic strategy firm 1 will likely have less demand when firm 2 enters the market suggests that a different strategy is necessary. Taking into account the competition from firm 2, a rational firm 1 would want to change its strategy to maximize its own expected profit. The result should be the Nash equilibrium described in Section 4. Finally, note that we also report the optimal Nash equilibrium prices.

### Table 1

<table>
<thead>
<tr>
<th>$x_2$</th>
<th>Nash equilibrium</th>
<th>Monopolistic strategy</th>
</tr>
</thead>
<tbody>
<tr>
<td>Firm 1</td>
<td>Firm 2</td>
<td>Firm 1</td>
</tr>
<tr>
<td>$\infty$</td>
<td>25264.73</td>
<td>100</td>
</tr>
<tr>
<td>0.25</td>
<td>23920.34</td>
<td>11741.2</td>
</tr>
<tr>
<td>0.5</td>
<td>23687.42</td>
<td>15105.1</td>
</tr>
<tr>
<td>0.75</td>
<td>21439.75</td>
<td>16267.08</td>
</tr>
<tr>
<td>1</td>
<td>19494.13</td>
<td>19494.13</td>
</tr>
<tr>
<td>1.5</td>
<td>16067.35</td>
<td>26748.09</td>
</tr>
<tr>
<td>2</td>
<td>13995.59</td>
<td>34413.47</td>
</tr>
<tr>
<td>2.5</td>
<td>12727.14</td>
<td>41336.15</td>
</tr>
</tbody>
</table>

* *b* Firm 1 ignores the competition and uses the monopolistic policy, and firm 2 maximizes its own expected profit, knowing firm 1 uses the monopolistic policy. These two columns show the percentages of expected profits under the monopolistic policy compared to the expected profits under the Nash equilibrium.

and one of them (firm 1) does not have access to the real-time information of the market stockpile while the other firm (firm 2) has the real-time information, and both firms know this fact. Firm 1 only knows the initial market stockpile, the market size, and the pricing history of both firms. Assuming that the initial market stockpile is public information is reasonable; for example, by introducing a new product one can assume that the initial market stockpile is zero at the beginning of the time horizon.

Let $p \equiv (p_1; p_2) = (p_1^0, p_1^1, \ldots, p_1^{2t-1}; p_2^0, p_2^1, \ldots, p_2^{2t-1})$ denote the price history charged by firm 1 and firm 2 since the beginning of time horizon until period $t - 1$. At the beginning of period $t$, firm 1 uses this pricing history and the initial market stockpile, $m_t$, to recursively estimate the market stockpile, $m_t^{2t-1}$, using the following procedure:

#### Procedure 1: Estimating market stockpile in period $t$ by firm 1.

1. Let $i = 0$ and $m_i = m_0$.

2. If $i = t$, return $m_t$.

3. Otherwise, calculate
   $$ m_i = m_{i-1} + N_i(m_{i-1})(1 - q_0) \left[ p_1^i(m_{i-1}) - C(m_{i-1}) \right] $$
   where $N_i(m_{i-1}) = N(m_{i-1})(1 - q_0)$. In other words, firm 1 solves Eq. (20) such that firm 2 uses the same pricing history.

   $$ p_1^i(m_{i-1}) = \max_{p_1} p_1^i(m_{i-1}) + U_2^i(m_{i-1}) $$

   for all $p_2 > 0$.

   where

   $$ m_{i-1} = m_{i-1} + N(m_{i-1})(1 - q_0) \left[ p_1^i(m_{i-1}) - C(m_{i-1}) \right] $$

   and $N_i(m_{i-1}) = N(m_{i-1})(1 - q_0)$. In other words, firm 1 solves Eq. (20) such that firm 2 uses the same pricing history.

4. Let $i = i + 1$ and go to Step 2.

6. A heuristic policy under incomplete-information

In Section 4, we studied a multiple period game assuming that both firms know the real-time market stockpile. In that case, at the beginning of each period, both firms observe the exact number of units in the channel and post their prices accordingly. Although this assumption is appealing for analytically calculating the equilibrium prices, in practice it may be very costly, or sometimes impossible, to determine the exact stockpile level. In this section, we provide a heuristic policy that allows us to relax this assumption for one firm. We develop a heuristic policy in which a firm only needs to know the initial stockpile level and price history but does not need to keep track of the real-time market stockpile level. This heuristic is simply based on estimating the average market stockpile using the MNL model described in Section 2.

Now consider the dynamic pricing problem in Section 4 from firm 1’s point of view. Firm 1 and firm 2 are no longer identical and one of them (firm 1) does not have access to the real-time information of the market stockpile while the other firm (firm 2) has the real-time information, and both firms know this fact. Firm 1 only knows the initial market stockpile, the market size, and the pricing history of both firms. Assuming that the initial market stockpile is public information is reasonable; for example, by introducing a new product one can assume that the initial market stockpile is zero at the beginning of the time horizon.

Let $p \equiv (p_1; p_2) = (p_1^0, p_1^1, \ldots, p_1^{2t-1}; p_2^0, p_2^1, \ldots, p_2^{2t-1})$ denote the price history charged by firm 1 and firm 2 since the beginning of time horizon until period $t - 1$. At the beginning of period $t$, firm 1 uses this pricing history and the initial market stockpile, $m_t$, to recursively estimate the market stockpile, $m_t^{2t-1}$, using the following procedure:

#### Procedure 1: Estimating market stockpile in period $t$ by firm 1.

1. Let $i = 0$ and $m_i = m_0$.

2. If $i = t$, return $m_t$.

3. Otherwise, calculate
   $$ m_i = m_{i-1} + N_i(m_{i-1})(1 - q_0) \left[ p_1^i(m_{i-1}) - C(m_{i-1}) \right] $$
   where $N_i(m_{i-1}) = N(m_{i-1})(1 - q_0)$. In other words, firm 1 solves Eq. (20) such that firm 2 uses the same pricing history.

   $$ p_1^i(m_{i-1}) = \max_{p_1} p_1^i(m_{i-1}) + U_2^i(m_{i-1}) $$

   for all $p_2 > 0$.

   where

   $$ m_{i-1} = m_{i-1} + N(m_{i-1})(1 - q_0) \left[ p_1^i(m_{i-1}) - C(m_{i-1}) \right] $$

   and $N_i(m_{i-1}) = N(m_{i-1})(1 - q_0)$. In other words, firm 1 solves Eq. (20) such that firm 2 uses the same pricing history.

7. Numerical study

In this section, we use a computational experiment to quantify the impact of different parameters of the model on the performance of different pricing policies. First, we study the performance

---

*Note that, since firm 1 ignores firm 2 in the monopolistic strategy, it charges a single price ($81.59) for all values of $x_2$. However its relative price levels increases (as in Table 3) since the price levels in equilibrium decreases.*
of different pricing policies for different values of \( \alpha_2 \). Table 2 reports the expected profit results of the example used in Table 1, with three cases of Nash equilibrium, Heuristic–Optimal, and Heuristic–Heuristic policies. Columns 2 and 3 report the results of the full information case when both firms use Nash equilibrium prices. In columns 4 and 5, we show the results when firm 1 uses the heuristic policy and firm 2 optimizes with full information. Finally the last two columns show the results when both firms have limited information and use the heuristic. Note again that, in the Nash equilibrium case, as the quality of the second product gets better, the equilibrium profit of firm 1 decreases while the equilibrium profit of firm 2 increases.

The price levels are reported in Table 3. The setting of this table is similar to that of Table 2 with the addition of extra columns to report price levels under the monopolistic strategy. Columns 2 and 3 provide the average equilibrium prices over all periods. Note that, to be able to compare the prices under the Nash equilibrium policy with prices under the other two policies, we need to use the average market stockpile under the heuristic policy over the time horizon. In more detail, let us assume that \( m^* \) is the average market stockpile at the beginning of period \( t \) using Procedure 1 that has been used to calculate the prices under heuristic policies. Then the corresponding average prices under the Nash equilibrium are \( E'(p_i(m^*)) \) for \( i = 1, 2 \), where \( E'(x^*) \) is the expected value \( x^* \) over \( t \).

For example, as seen in Table 3, when \( \alpha_2 = 0.25 \), then 72.2 and 60.6 are \( E'(p_1(m^*)) \) and \( E'(p_2(m^*)) \), respectively. Other columns in Table 3 show the percentages of average prices over the sale horizon under different solution strategies.

As expected, note from Table 2 that, in the Nash equilibrium case, as the quality of firm 2’s product gets better the equilibrium profit of firm 1 decreases while equilibrium profit of firm 2 increases. The magnitude of these changes is interesting. For instance, as \( \alpha_2 \) increases from 0.25 to 0.5, firm 1’s profit decreases only slightly (less than $300), while firm 2’s profit increases quite substantially (more than $3000). A weak entrant does not hurt the entrenched firm 1, even though the entrant (firm 2) makes large gains. As firm 2’s quality approaches parity with firm 1, as seen by \( \alpha_2 \) increasing from 0.25 to 1.0, firm 1’s profit decreases by about $4000, while firm 2’s profit increases by over $7000. Firm 1 is now hurt much more substantially because of a stronger competitor, even as that competitor’s profit grows rapidly. Now consider the effect of a dramatic increase in firm 2’s quality, such that \( \alpha_2 \) reaches 5.00 – a dominant quality position. Firm 1 still retains some profit, although less than half of its profit when \( \alpha_2 \) was 1.00. But firm 2’s profit has increased fivefold. Perhaps most interesting is the fact that total industry profit increases as firm 2’s quality increases, although not monotonically. The overall picture is that balanced quality is good for consumers by driving competition in prices, and yet it decreases industry profits.

Further insight on these cases can be seen from Table 3. Of course, in the equilibrium case, firm 2 charges higher prices for higher values of \( \alpha_2 \). The market will bear higher prices for higher quality products. When \( \alpha_2 \) equals 1.00, both firms charge the same price, as the model would predict. As the quality of firm 2 gets still higher, firm 1 must charge lower prices to attract customers, and hence it receives lower profits (Table 2). Yet, notice that as \( \alpha_2 \) increases from 2.5 to 5.0, firm 2’s price only increases marginally. It would seem that the market cannot bear prices beyond a certain level, even with extremely high quality relative to firm 1.

Now consider the Heuristic–Optimal case, and note that as expected firm 2’s expected profit is always higher than firm 1’s profit. Firm 1’s suboptimal approach is due to the fact that it does not know the real-time market stockpile and must estimate the market stockpile using Procedure 1 and thus use the heuristic policy. On the other hand, firm 2 uses the real-time information and mathematical programming in Eq. (21) to find the optimal price levels. In this case, firm 1 charges higher prices compared to the equilibrium policy (Table 3, column 4), which is due to the accumulated errors in estimation of the market stockpile over the time horizon.

Note that, in the cases when \( \alpha_2 \) is less than 1.0, firm 1 charges significantly higher prices (up to 10% of the equilibrium price levels). In response, firm 2, which observes that firm 1 charges higher prices, also charges higher prices. This allows firm 2 to generate several percentage points higher profit (up to 10% of equilibrium profit). When \( \alpha_2 \) increases above 1.0, firm 1 continues to lower its prices until at \( \alpha_2 = 5.0 \), firm 1’s price is only 1.73% higher than its equilibrium price levels. In response, firm 2 also charges lower prices and yet even generates higher profit (up to 10% of the expected profit, Table 2, column 5 and Table 3, column 5).

In the third case, both firms have limited information and use the heuristic policy. Even without knowing the real-time market stockpile, both firms attain at least 97% of the revenue in the complete-information model (Table 2, columns 6 and 7). These results are not surprising because our heuristics provide a good approximation to the real market stockpile, so the firms best response is close to the prices in the Nash equilibrium. Note that, under the heuristic policy, a firm can generate more profit than it can in the Nash equilibrium. For instance, in Table 2, column 7 (when
The performance of the heuristic policy decreases for higher values of \( \gamma_2 \). This is due to the fact that the equilibrium profit is an increasing function with respect to \( \gamma_2 \). We can verify a well-known rule in the dynamic pricing literature that a firm can achieve by not ignoring competition from other firms and by avoiding using a strategy as if it enjoyed monopoly in the market. Although, in the main body of the paper we have assumed that the market stockpile can be observed by the firms, we have relaxed this assumption in Section 6 by introducing a heuristic for a game with limited information. We consider two cases where either one firm has limited information and the other firm has full information (Heuristic–Optimal) or both firms have limited information (Heuristic–Heuristic). Using numerical analysis, we illustrate the impact of different parameters of the model such as product quality and remaining time horizon on the performance of our heuristic policy.

### Table 4

<table>
<thead>
<tr>
<th>( T )</th>
<th>Equilibrium ($)</th>
<th>Heuristic vs. Optimal</th>
<th>Heuristic vs. Heuristic</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Firm 1</td>
<td>Firm 2</td>
<td>Firm 1</td>
</tr>
<tr>
<td>10</td>
<td>4215.92</td>
<td>2802.99</td>
<td>99.54</td>
</tr>
<tr>
<td>20</td>
<td>9277.93</td>
<td>5972.85</td>
<td>99.01</td>
</tr>
<tr>
<td>30</td>
<td>13479.94</td>
<td>9539.83</td>
<td>99.01</td>
</tr>
<tr>
<td>40</td>
<td>18912.62</td>
<td>12113.51</td>
<td>98.87</td>
</tr>
<tr>
<td>50</td>
<td>23687.42</td>
<td>15105.10</td>
<td>97.92</td>
</tr>
<tr>
<td>75</td>
<td>38013.26</td>
<td>21263.43</td>
<td>95.1</td>
</tr>
<tr>
<td>100</td>
<td>47630.06</td>
<td>28490.41</td>
<td>94.12</td>
</tr>
</tbody>
</table>

- \( \gamma_2 = 2, 2.5, \) and \( 5 \), firm 2 generates more profit using heuristic policy than it did under the equilibrium policy. In these cases, Table 3, column 7 illustrates that firm 2 charges lower prices compared to equilibrium prices, which attracts more customers to the market, and in turn generates higher profit for firm 2. Once again, exceptional quality, relative to the competition, allows a firm to suboptimize its pricing decisions and still make reasonable profits. If quality is relatively comparable between the two firms, say \( \gamma_2 \) between 0.5 and 1.5, the heuristic policies hurt profitability for both firms, as both firms charge higher than optimal prices (see Tables 2 and 3, columns 6 and 7).

In comparing the Heuristic–Optimal and Heuristic–Heuristic cases for firm 2, one can observe that firm 2 charges lower prices and generates lower profits when it uses the heuristic. See Tables 2 and 3, columns 5 and 7. This is as expected: regardless of relative quality level, if firm 1 uses a heuristic, firm 2 will benefit from the effort to optimize prices using the additional information available to it.

In Table 4, we study the impact of the remaining time horizon on the performance of different pricing policies. The setting of Table 4 is similar to that of Table 2 where \( T \) takes on values from 10 to 100, \( \gamma_2 = 0.5 \), and other parameters remain the same. In this table, we can verify a well-known rule in the dynamic pricing literature that the equilibrium profit is an increasing function with respect to remaining time. Another interesting observation is that the performance of the heuristic policy decreases for higher values of \( T \). Considering the Heuristic–Optimal case (columns 4 and 5), more remaining time causes firm 1 to decrease profit by using the heuristic while it helps firm 2 to make more profit. This is due to the increased accumulated error of estimating the market stockpile as \( T \) increases. In this case, firm 2 takes advantage and receives more market share and thus more profit. With the same reasoning, when both firms use the heuristic policy, they both lose profit for larger values of \( T \).

### 8. Conclusions

In this paper, we propose a game-theoretic model to study the dynamic pricing competition between two firms competing in a market that the products can be stored for future use. A discounted price in any period causes the customers to buy more than needed in order to store for future use when the discounted price might not be available. Therefore, a market stockpile of products is formed that has direct impact on total demand. We developed our model utilizing an MNL model in a finite horizon. From a theoretical point of view, the most important finding of this paper is Theorem 2 where we show that there exists a unique Nash equilibrium in pure strategies for the multi-period game. In addition, we show, by numerical examples, the magnitude of the benefits firms can achieve by not ignoring competition from other firms and by avoiding using a strategy as if it enjoyed monopoly in the market.

### References


